Stellar Evolution 2016 Q1

Ugeseddel 3 (week 37)

In the lecture Thursday September 15 Günter will finish the discussion of the numerical treatment of stellar evolution (*Kippenhahn, Weigert & Weiss*, Chapter 12). On Monday 19 September he begins the discussion of the detailed treatment of the equation of state of stellar matter, including partial ionization and degeneracy (*Kippenhahn, Weigert & Weiss*, Chapters 13 – 16). This will be completed in the following lectures (22 and 26 September), together with other aspects of the physics of stellar interiors, i.e, opacity (*Kippenhahn, Weigert & Weiss*, Chapter 17) and energy generation (*Kippenhahn, Weigert & Weiss*, Chapter 18).

The exercise class on 21 September will consider:

i) Exercise U3.1 below.

i) Exercise U3.2 below. (This will likely be continued at the exercise classes the following week.)

Correction to ***Kippenhahn, Weigert & Weiss*****:**

- **p. 137, line below Eq. (14.43)**: Here there is a slightly confusing misplaced comma. The correct piece of text is: '. . . is the Bohr radius, ν the quantum number, and n_{H} the . . . '.

14 September 2016 Jørgen Christensen-Dalsgaard

Exercise U3.1:

The convective speed, Mach number and turbulent pressure. Here we consider the behaviour of the speed v of the convective elements in the different limits discussed by *Kippenhahn, Weigert & Weiss*, Section 7.3. Particularly important is the Mach number $\mathcal{M} = v/c_s$, where c_s is the sound speed, given by

$$c_s^2 = \frac{\gamma P}{\rho} \; .$$

i) Show, from *Kippenhahn, Weigert & Weiss* eq. (7.6) that

$$\mathcal{M}^2 = \alpha_{\rm m}^2 (\nabla - \nabla_{\rm e}) \frac{\delta}{8\gamma} \; , \qquad (2.1)$$

if, as usual, we assume that the mixing length is given by $l_{\rm m} = \alpha_{\rm m} H_P$ where $\alpha_{\rm m}$ is a constant of order unity.

ii) Estimate where in a star $\mathcal{M}$ is likely to be large. The figure below, for a model of the present Sun, can be used to estimate the variation of U and W throughout the solar convection zone. For these estimates you can assume that $\delta = 1$, $\gamma = 5/3$ and $\alpha_{\rm m} = 1.9$.

iii) When the Mach number is close to one, the momentum transport from the convective elements affects the hydrostatic balance. This is characterized by the *turbulent pressure* $P_{\rm turb} = \rho v^2$, which should be included in the equation of hydrostatic equilibrium. Show that

$$\frac{P_{\rm turb}}{P} = \gamma \mathcal{M}^2 \; .$$

iv) To take a closer look at the properties of convection, read the data on which the figure is based, from
`http://astro.phys.au.dk/~jcd/stel-struc/convq.l5bi.d.15`
and make your own plots of U and W, using Matlab or another suitable language. A convenient abscissa is $\log P$.

v) It is convenient to work in terms of *Kippenhahn, Weigert & Weiss* (KWW) Eq. (7.26). To get a first estimate of $x = \nabla - \nabla_{\rm ad}$ make the *ansatz* that $U^2 \ll x \ll W$ and show that

$$x \simeq x_0 \equiv \left(\frac{8}{9} UW\right)^{2/3} \; .$$

vi) Make plots of $\nabla - \nabla_{\rm ad}$, $\mathcal{M}$ and $P_{\rm turb}/P$ using this solution. You may assume that $\nabla_{\rm e} = \nabla_{\rm ad}$ (but you can of course also check to what extent this is satisfied).

vii) With this solution, check the validity of the inequalities assumed in v).

viii) KWW, Eq. (7.26) can be solved using Newton iteration, as probably discussed by Günter. Define

$$f(x) = [\sqrt{x + U^2} - U]^3 + \frac{8}{9} U (x - W) ,$$

and show that, given the estimate x_0, an improved solution can be obtained as

$$x = x_0 - \frac{f(x_0)}{({\rm d}f/{\rm d}x)|_{x=x_0}} .$$

Compute an improved $x = x_1$ using this expression. The procedure can obviously be iterated until the change in x is negligible. Do that.

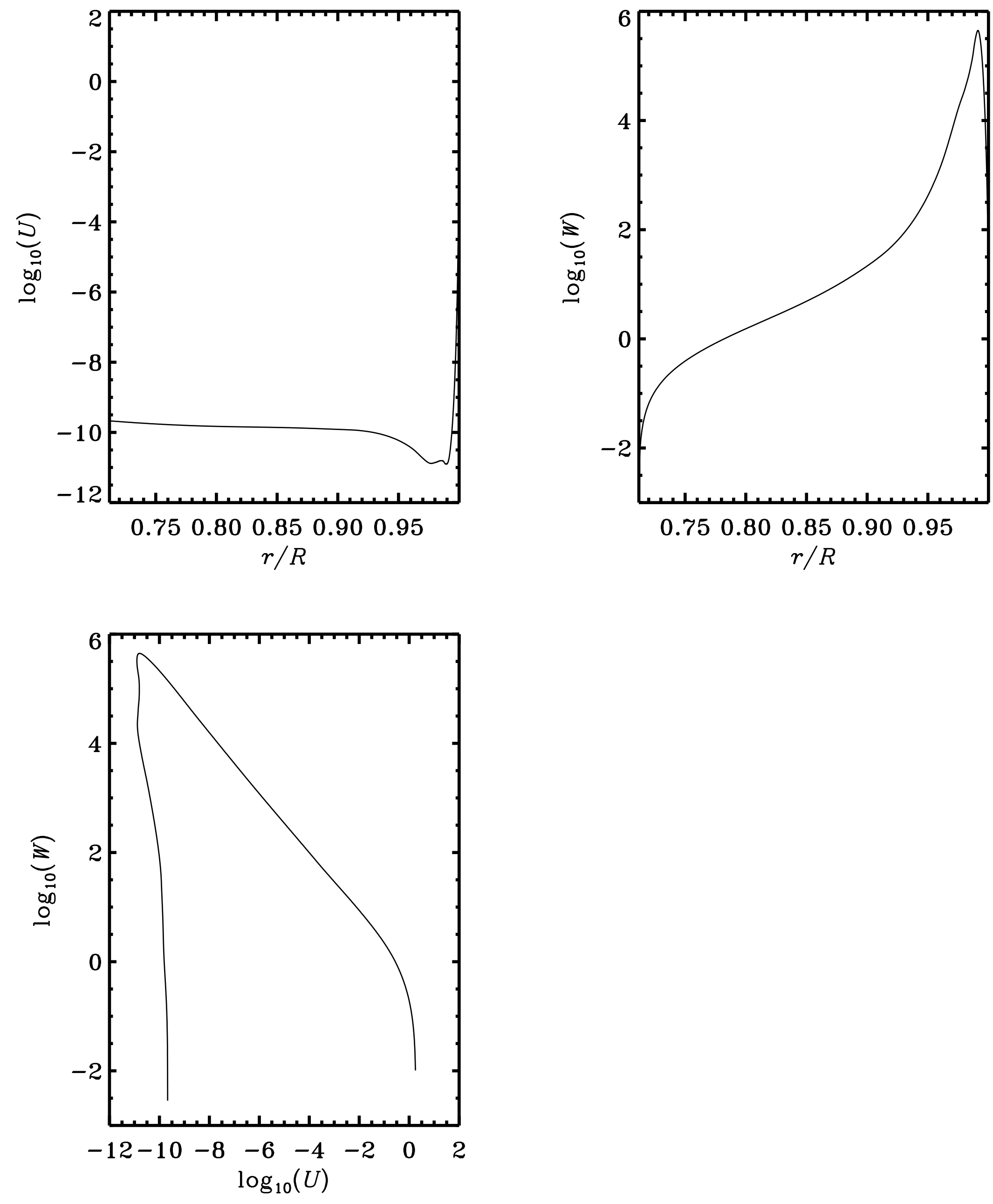

Figure 1: Relevant quantities for Exercise U3.1 above. Note that the data can also be found in
`http://astro.phys.au.dk/~jcd/stel-struc/convq.l5bi.d.15`.

Exercise U3.2:

Solution of a boundary-value problem with the Henyey method. The equations of stellar structure form a *boundary-value problem*, with conditions imposed at both ends of the interval (the stellar centre and surface). This exercise provides a very simple example of the technique that is generally used to solve these equations. Consider the simple boundary value problem on the interval $[0, 1]$:

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}x} &= y_2 \ , \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} &= -y_1 \ , \end{aligned}$$

$$y_1(0) = 1 \ , \qquad y_1(1) = 0 \ .$$

Introduce a uniform mesh $0 = x^1, x^2, \ldots, x^N = 1$ in the independent variable x, and consider the following difference approximation of the differential equation:

$$\begin{aligned} \frac{y_1^{j+1} - y_1^j}{h} &= {}^1\!/\!_2(y_2^{j+1} + y_2^j) \ , \\ \frac{y_2^{j+1} - y_2^j}{h} &= -{}^1\!/\!_2(y_1^{j+1} + y_1^j) \ ; \end{aligned}$$

As usual, $y_i^j \equiv y_i(x^j)$, and $h = x^{j+1} - x^j$.

i) Show that the solution satisfies

$$\left\{ \begin{array}{c} y_1^{j+1} \\ y_2^{j+1} \end{array} \right\} = \mathcal{H} \left\{ \begin{array}{c} y_1^j \\ y_2^j \end{array} \right\} \ ,$$

where the matrix $\mathcal{H}$ is given by

$$\mathcal{H} = \left\{ \begin{array}{cc} \dfrac{1 - {}^1\!/\!_4 h^2}{1 + {}^1\!/\!_4 h^2} & \dfrac{h}{1 + {}^1\!/\!_4 h^2} \\ -\dfrac{h}{1 + {}^1\!/\!_4 h^2} & \dfrac{1 - {}^1\!/\!_4 h^2}{1 + {}^1\!/\!_4 h^2} \end{array} \right\} \ .$$

ii) Hence show that the solution at $x = 1$ is related to the solution at $x = 0$ by

$$\left\{ \begin{array}{c} y_1(1) \\ y_2(1) \end{array} \right\} = \mathcal{H}^{N-1} \left\{ \begin{array}{c} y_1(0) \\ y_2(0) \end{array} \right\} \ .$$

iii) How may this be used to determine $y_2(0)$, from the boundary condition at $x = 1$?

iv) Show that, once $y_2(0)$ has been found, the solution is determined everywhere.

v) Write a small programme to carry out this solution, in Matlab or another suitable language. Compare with the analytical solution (find it!) and check how the error changes with N.
