Boundary conditions

- ODE of 4th order, describing the spatial stellar equations at fixed time t, is solved as a boundary-value problem (BVP). A 4th - order ODE system requires therefore 4 boundary conditions (BCs).

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi r^2 \varrho} \; ,$$
$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \; ,$$
$$\frac{\mathrm{d}l}{\mathrm{d}m} = \varepsilon_{\mathrm{n}} - \varepsilon_\nu \; ,$$
$$\frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{GmT}{4\pi r^4 P}\nabla \; ,$$
$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) \; , \quad i = 1, \ldots, I \; .$$

BCs are split: some given at the centre (m=0) and some near the surface (m_F <= M).

Central BCs are simple, whereas surface BCs are much more involved.

BCs do affect solutions of ODEs rather severly.

In the following discussion we assume complete equilibrium.

Boundary conditions: central BC

- central BC are defined at m = 0.
- because density must not vanish and stay finite (no cavity or singularity at m = 0), we have r = 0.
- and because ε must remain finite, luminosity l = 0 at centre.

Central BC at $m = 0: \quad r = 0, \quad l = 0 \; .$

- nothing is known about the central temperature T_c and pressure P_c.
- without outer BCs, i.e. using only the two central BCs m = 0 and l = 0 we still allow a **two-parameter set of solutions** by integrating outwards from r = 0, l = 0 and arbitrary values for P_c and T_c.

Boundary conditions: central BC

Let's consider functional behaviour of r, l, P and T near centre ($m \to 0$):

from continuity eq. $\quad d(r^3) = \frac{3}{4\pi\varrho} dm \; ,$

and integrating with constant ρ_c (i.e. m<<, r<<) leads to

$$r = \left(\frac{3}{4\pi\varrho_c}\right)^{1/3} m^{1/3} \; .$$

Similar integration of energy eq. leads to

$$l = (\varepsilon_{\mathrm{n}} - \varepsilon_\nu + \varepsilon_{\mathrm{g}})_{\mathrm{c}} \; m \; .$$

Note that central BCs lead to vanishing integration constants.

From hydrostatic equation using r-expression for small m leads to

$$\frac{dP}{dm} = -\frac{G}{4\pi}\left(\frac{4\pi\varrho_c}{3}\right)^{4/3} m^{-1/3} \; ,$$

Boundary conditions: central BC

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$$\frac{dP}{dm} = -\frac{G}{4\pi}\left(\frac{4\pi\varrho_c}{3}\right)^{4/3} m^{-1/3} \; ,$$

which, after integration gives

$$P - P_c = -\frac{3G}{8\pi}\left(\frac{4\pi}{3}\varrho_c\right)^{4/3} m^{2/3} \; .$$

The pressure gradient dP/dr =0 at centre, which can be seen from hydrostat. eq.

$$\frac{dP}{dr} \sim \frac{m}{r^2} \sim \frac{r^3}{r^2} \to 0 \qquad \text{for } r \to 0.$$

Boundary conditions: central BC

The variation of T near the centre is considered first for the radiative case, for which we have

$$\frac{dT}{dm} = -\frac{3}{64\pi^2 ac}\frac{\kappa l}{r^4 T^3} .$$

With $P \rightarrow P_c$, $T \rightarrow T_c$ opacity $\kappa \rightarrow \kappa_c$, and replacing l and r by their central approximations, followed by integration, leads to

$$T^4 - T_c^4 = -\frac{1}{2ac}\left(\frac{3}{4\pi}\right)^{2/3}\kappa_c(\varepsilon_n - \varepsilon_\nu + \varepsilon_g)_c \varrho_c^{4/3} m^{2/3} \text{ (radiative)} .$$

For the (adiabatic) convective case we replace ∇ by ∇_{ad} in the transport equation (dT/dm) and replace r by its central approximation. Subsequent integration yields

$$\ln T - \ln T_c = -\left(\frac{\pi}{6}\right)^{1/3} G \frac{\nabla_{ad,c}\varrho_c^{4/3}}{P_c} m^{2/3} \text{ (convective)} .$$

Boundary conditions: surface BC

Pressure above this layer is with $g=g_0=GM/R^2$ = constant

$$P_{r=R} = \int_R^\infty g\varrho\, dr = g_0 \int_R^\infty \varrho\, dr ,$$

and replace density integral by the definition of the optical depth τ, i.e. by $\tau/\bar{\kappa}$, we find for $\tau = 2/3$

$$P_{r=R} = \frac{GM}{R^2}\frac{2}{3}\frac{1}{\bar{\kappa}} .$$

The temperature at the photosphere is defined to be that of a "**black radiator**", i.e.

$$L = 4\pi R^2 \sigma\, T_{eff}^4 ,$$ BC1 @surface

where $\sigma = ac/4$ is the Stefan-Boltzmann constant.

Boundary conditions: surface BC

Strict surface conditions are very involved. ⟶ Approximations are typically employed.

(a) The most naïve "zero conditions":

$$m \rightarrow M : \quad P \rightarrow 0, \quad T \rightarrow 0 .$$

(b) Let's define a stellar "surface" that defines the location $r=R$, the so-called "photosphere", from which most of the radiation escapes into space.
It is the location where the optical depth τ of the overlaying layers is $\tau = 2/3$,

$$\tau := \int_R^\infty \kappa\varrho\, dr = \bar{\kappa}\int_R^\infty \varrho\, dr .$$

Pressure above this layer is with $g=g_0=GM/R^2$ = constant :

$$P_{r=R} = \int_R^\infty g\varrho\, dr = g_0 \int_R^\infty \varrho\, dr ,$$

Boundary conditions: surface BC

In the definition of the photosphere we assumed an average value $\bar{\kappa}$ for the opacity. With detailed opacity values at hand we can use $\kappa(P,T)$ instead and adopt for the T-dependence the so-called Eddington approximation

$$T^4(\tau) = \frac{3}{4}\left(L/4\pi R^2\sigma\right)\left(\tau + \frac{2}{3}\right),$$

which obviously results in $T = T_{eff}$ for $\tau = 2/3$.

The differential form for the definition of the optical depth τ is

$$dr/d\tau = -1/(\kappa\varrho),$$

and using $dP/dr = -g\varrho$ we obtain

$$\frac{dP}{d\tau} = \frac{Gm}{r^2\kappa} ,$$

which is to be integrated from $\tau = 0$ to $\tau = 2/3$ with the **BC $P(\tau = 0) = 0$**.

Boundary conditions: surface BC

$$\frac{dP}{d\tau} = \frac{Gm}{r^2\kappa} \simeq \frac{GM}{R^2}\kappa^{-1}(P(\tau),T(\tau)) \qquad \text{BC2 @surface}$$

For g on the right-hand-side of this equation we can safely assume $g=GM/R^2$, and the T-dependence of $\kappa(P,T)$ is obtained from the Eddington approximation $T^4(\tau)$.

↓

improved value for P_R.

This approach is called the Eddington grey atmosphere.

Severest assumption in adopting these surface BCs (BC1 & BC2) is the diffusion approximation to radiative transfer, i.e. dT/dm in stellar structure equations, which only holds in optically thick (τ>>) regions, where here we defined the surface BCs at $\tau = 2/3$.

Other approaches are recommended, such as fitting the stellar interior solution to a separately computed atmosphere model, which includes more involved physics.

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Transition between interior and outer, atmospheric, solution is done at certain mass value m_F, the "fitting mass", which should be located far enough below photosphere, where the stratification is optically thick (τ>>), but still as close as possible to M.

interior solution: with given X_i and M (and central BCs) we obtain a two-parameter set of possible atmospheric solutions, i.e. R and T_{eff}, or R and L.

↓ integrating down to m_F to obtain "exterior" values

note $l = L$ = const.

exterior solution: $\boxed{r = r_F^{ex}, P = P_F^{ex}, T = T_F^{ex}, l = l_F^{ex} = L.}$

↓

matching interior solution: The outer BCs @ m_F require one quartet $r_F^{in}, \ldots, l_F^{in}$ to match the atmospheric exterior solutions $r_F^{ex}, \ldots, l_F^{ex}$:

Boundary conditions: surface BC

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fit condition: $r_F^{ex} = r_F^{in}, \quad P_F^{ex} = P_F^{in}, \quad T_F^{ex} = T_F^{in}, \quad l_F^{ex} = l_F^{in} \quad .$

These fits are possible because interior solution has 2 degrees of freedom (P_c and T_c) and so does the exterior solutions (R and L).

The exterior (atmospheric) solutions yield @ m_F the following four functions:

$$r_F^{ex}(R,L), P_F^{ex}(R,L), T_F^{ex}(R,L), l_F^{ex}(R,L).$$

well behaved ↓ we can invert it

$R = R(r_F^{ex}, L)$

$l_F^{ex} = L$ ... because ε_n=0 in surface layers

⬇

$$P_F^{ex}(R(r_F^{ex},L),L) := \pi(r_F^{ex},L) \; ,$$
$$T_F^{ex}(R(r_F^{ex},L),L) := \theta(r_F^{ex},L) \; .$$

Boundary conditions: surface BC

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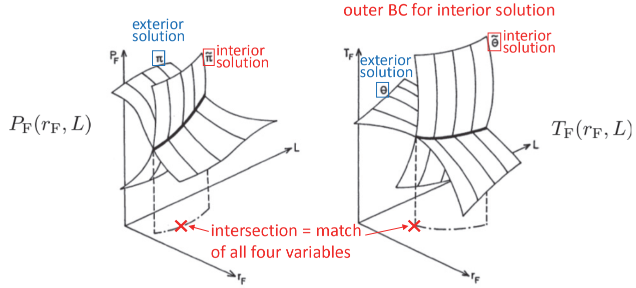
For any pair $r_F^{ex}, L,$ the π and θ functions provide values for P_F^{ex} and T_F^{ex} .

From the fit conditions we have $\boxed{P_F^{in} = \pi(r_F^{in},L), \quad T_F^{in} = \theta(r_F^{in},L)}$.

outer BC for interior solution



Boundary conditions: effect of surface conditions and properties of envelope solutions

- we consider stars in complete (mechanical and thermal) equilibrium.

- note: l and m are essentially constant over large r-range ($\varepsilon = 0$, ρ<<)
  (Sun: only 10% of M outside R/2).

Radiative envelopes:

Because m varies so little we use P as the independent variable instead of m.
The radiative transport eq. then becomes:

$$\frac{\partial T}{\partial P} = \frac{3}{64\pi\sigma G}\frac{\kappa l}{T^3 m} \qquad (\sigma = ac/4).$$

The (mean) opacity is *approximated* by the power-law (Kramers opacity)

$$\kappa = \kappa_0 P^a T^b\,, \qquad \kappa_0 = \text{constant}, \qquad \text{typically } a > 0, b < 0.$$

$$\longrightarrow \quad \frac{T^{3-b}}{P^a}\frac{\partial T}{\partial P} = \frac{3\kappa_0}{64\pi\sigma G}\frac{l}{m}\,.$$

Boundary conditions: effect of surface conditions and properties of envelope solutions

$$\frac{T^{3-b}}{P^a}\frac{\partial T}{\partial P} = \frac{3\kappa_0}{64\pi\sigma G}\frac{l}{m}\,.$$

With the *assumptions* $l \simeq L$ and $m \simeq M$ the r.h.s. becomes constant and $\int$ leads to

$$T^{4-b} = B(P^{1+a} + C)\,, \qquad \text{where } C \text{ is a constant of integration}$$

and

$$B = \frac{4-b}{1+a}\frac{3\kappa_0}{64\pi\sigma G}\frac{L}{M} > 0\,.$$

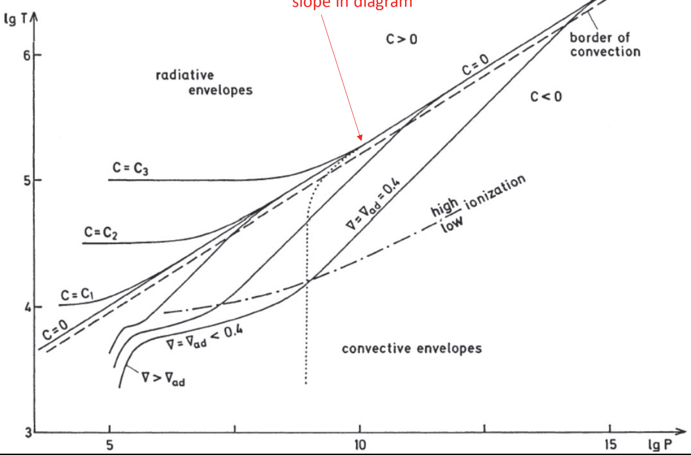
Typical values (e.g. for Sun) for the exponents in κ are a=1, b=-4.5 (= Kramers opacity)

$$\longrightarrow \quad T^{8.5} = B(P^2 + C)\,, \qquad B > 0\,.$$

approximate solution(s) for
stellar envelops with moderate T

Boundary conditions: effect of surface conditions …

$\boxed{T^{8.5} = B(P^2 + C)}$, $\quad \underbrace{\nabla \equiv d\ln T/d\ln P}_{\text{slope in diagram}} \longrightarrow \nabla = 0.235\dfrac{BP^2}{T^{8.5}}\,.$



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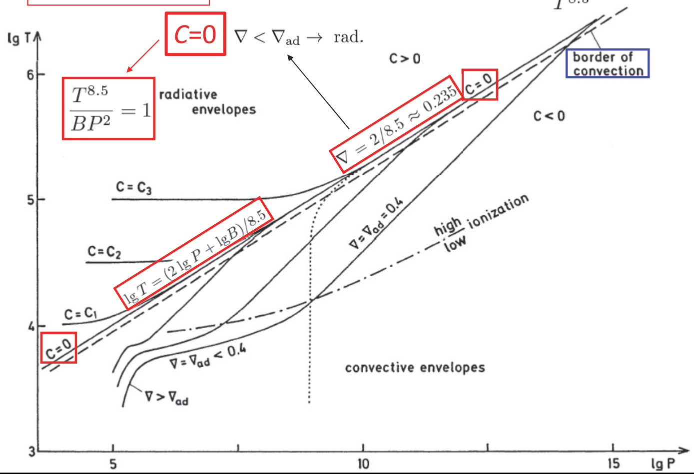

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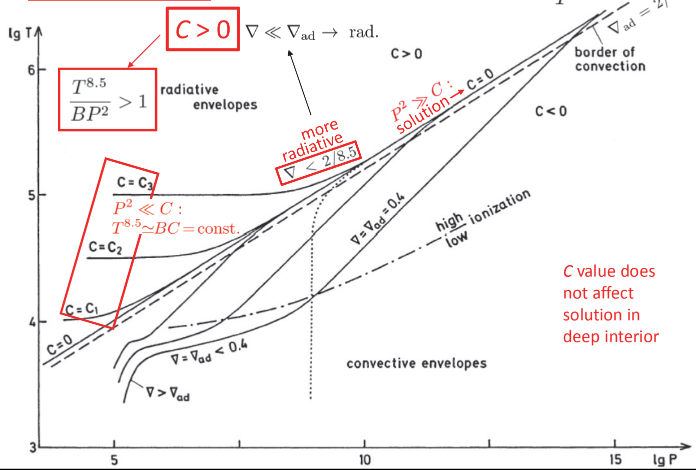
$\boxed{C > 0}$ $\nabla \ll \nabla_{\rm ad} \to$ rad.

$\boxed{\frac{T^{8.5}}{BP^2} > 1}$



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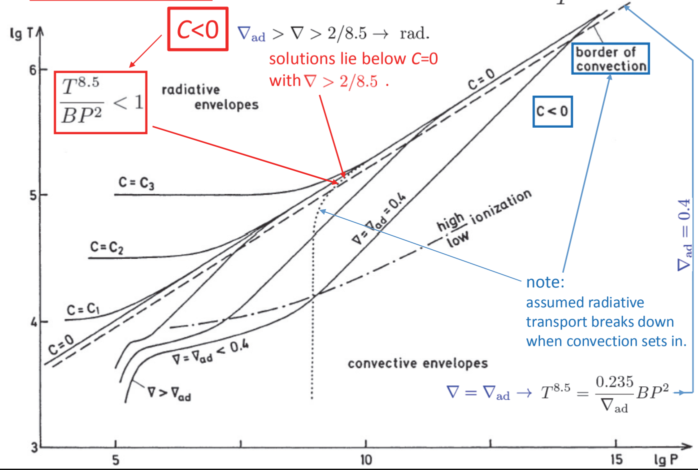
$\boxed{C<0}$ $\nabla_{\rm ad} > \nabla > 2/8.5 \to$ rad.

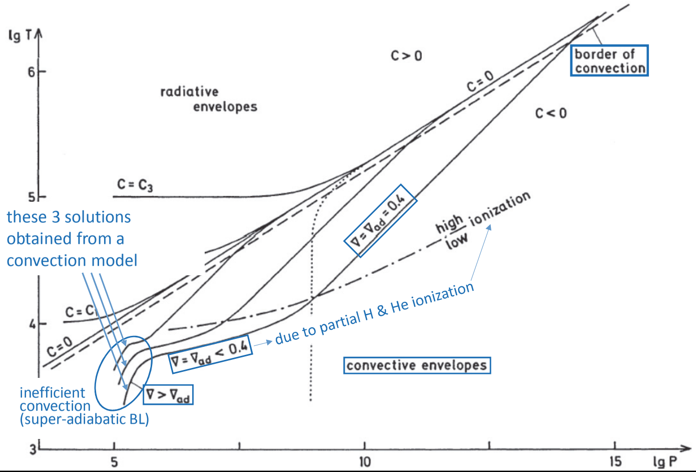
solutions lie below *C*=0 with $\nabla > 2/8.5$.

$\boxed{\frac{T^{8.5}}{BP^2} < 1}$

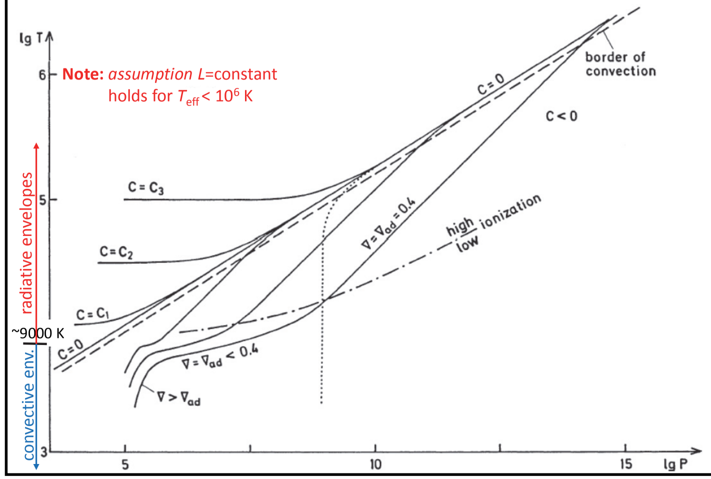


Boundary conditions: effect of surface conditions ...

Convection:



Boundary conditions: effect of surface conditions ...



Boundary conditions: the $T - r$ relation

Sometimes it is useful to know $T=T(r)$ just below the photosphere where $T=T_{\rm eff}$.

- from $$dP = -\frac{Gm}{r^2}\varrho\, dr = Gm\varrho\, \mathrm{d}\left(\frac{1}{r}\right)$$

and using the EOS for a perfect gas $T\varrho/P = \mu/\Re$ we obtain from $dT = T\nabla dP/P$

$$dT = \nabla \frac{G\mu}{\Re} m\ \mathrm{d}\left(\frac{1}{r}\right) .$$

- as before we *approximate* $m=M$ in the outer stellar layers and also *assume* ∇ = const.
  We then obtain from integration between points 1 and 2:

$$T_1 - T_2 = \nabla \frac{GM\mu}{\Re}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) .$$

- With point 2 defining the photosphere, i.e. $T_2=T_{\rm eff}$ and $r_2=R$ we have:

$$T - T_{\rm eff} = f\left(\frac{R}{r} - 1\right), \quad f = \nabla \frac{g\mu}{\Re}\frac{M}{R} .$$

- E.g.: for $M = M_\odot, R = R_\odot$, C=0 → $\nabla = 0.235$ and $\mu = 1$ we find $f = 5.4 \times 10^6$ K.

→ such large f leads to rapid increase of T, i.e. within 2% r: $T=10^5$K; @r/R=.8 (m/M=.99): $T=10^6$K.