The overall problem: The ODEs of stellar evolution

- describe conservation of mass, momentum and energy, transport of energy flux and change of chemical composition:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} ,$$
$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \partial^2 r/\partial t^2 (4\pi r^2)^{-1}$$
$$\frac{\partial l}{\partial m} = \varepsilon_{\rm n} - \underbrace{\varepsilon_\nu}_{\text{neutrino losses}} - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\varrho}\frac{\partial P}{\partial t} , \qquad \delta := -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P$$
$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla , \qquad \nabla := \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}$$
$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) , \quad i = 1, \ldots, I .$$

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$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla ,$$

in radiation zone: $\nabla := \nabla_{\rm rad} = \frac{3}{16\pi a c G}\frac{\kappa l P}{m T^4}$

in convection zone: ∇ from cubic equation (ML-model); $\nabla \simeq \nabla_{\rm ad} = P\delta/\rho c_p T$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) , \quad i = 1, \ldots, I .$$

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$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} , \qquad \frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} ,$$

Mechanical equation: coupled to the thermo-energetic equations only through ρ.

$$\frac{\partial l}{\partial m} = \varepsilon_{\rm n} - \varepsilon_\nu - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\varrho}\frac{\partial P}{\partial t} , \qquad \frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla ,$$

Thermo-energetic equations.

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right)$$

Chemical part: if $\tau_{\rm n} >>$, eqs can be decoupled from other 4 eqs: $X_i(m,t) \rightarrow X_i(m)|_t$.

The overall problem: The ODEs of stellar evolution

- structure equations contain material property functions:

$\varrho = \varrho(P, T, X_i)$, ... equation of state (EOS)

$$\left.\begin{aligned} c_P &= c_P(P,T,X_i) ,\\ \delta &= \delta(P,T,X_i) ,\\ \nabla_{\rm ad} &= \nabla_{\rm ad}(P,T,X_i) ,\end{aligned}\right\} \text{other thermodynamic (TD) quantities provided by EOS}$$

$\kappa = \kappa(P, T, X_i)$, ... (Rosseland) mean opacity

$$\left.\begin{aligned} r_{jk} &= r_{jk}(P,T,X_i) ,\\ \varepsilon_{\rm n} &= \varepsilon_{\rm n}(P,T,X_i) ,\\ \varepsilon_\nu &= \varepsilon_\nu(P,T,X_i) .\end{aligned}\right\} \text{nuclear transmutation quantities (reaction and energy rates), and enery loss due to neutrinos.}$$

- note: all functions depend on all types of nuclei X_i ($i = 1, \ldots, I$).

The overall problem: The ODEs of stellar evolution

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$$\frac{\partial l}{\partial m} = \varepsilon_\mathrm{n} - \varepsilon_\nu \underbrace{- c_P \frac{\partial T}{\partial t} + \frac{\delta}{\varrho}\frac{\partial P}{\partial t}}_{-T\frac{\partial s}{\partial t}}\,,$$
$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla\,,$$
$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right)$$

Nr. of equations : 4 + I
Nr. of variables : 4 + I
Independent variables: m, t

For $\dot M = 0$ we seek solutions for interval ($M = M(t_0)$) :

$$0 \le m \le M, \quad t \ge t_0\,.$$

We have a system of nonlinear, partial differential equations (PDE), which require boundary conditions (BC) and initial conditions (IC) at t=t_0.

IC: $r(m,t_0), \dot r(m,t_0), s(m,t_0)$, and $X_i(m,t_0)$.

The ODEs of stellar evolution: timescales & simplifications

- There are 3 types of time derivatives with associated timescales.

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho}\,,$$
$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}\,, \longrightarrow \tau_\mathrm{hyd}$$
$$\frac{\partial l}{\partial m} = \varepsilon_\mathrm{n} - \varepsilon_\nu - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}\,, \longrightarrow \tau_\mathrm{KH}$$
$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla\,,$$
$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right)\,, \quad i = 1,\ldots,I\,. \longrightarrow \tau_{X_i} = \tau_\mathrm{n}$$

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$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla\,,$$
$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right)\,, \quad i = 1,\ldots,I\,.$$

if $\begin{cases} \tau_\mathrm{hyd} \ll \tau_\mathrm{KH} \\ \tau_\mathrm{hyd} \ll \tau_\mathrm{n} \end{cases}$ $\longrightarrow$ star evolves along sequence of hydrostatic-equilibrium states with initial conditions $s(m,t_0)$ and $X_i(m,t_0)$ to be specified.

entropy s :

$$-c_P\frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t} = -T\frac{\partial s}{\partial t}$$

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if star evolves on timescale $\tau_\mathrm{KH} \ll \tau_\mathrm{n}$, then the time derivatives in energy eq. can also be neglected.

The ODEs of stellar evolution: timescales & simplifications

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$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} , \quad \frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} , \quad \frac{\partial l}{\partial m} = \varepsilon_\mathrm{n} - \varepsilon_\nu , \quad \frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla ,$$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) , \quad i = 1, \ldots, I .$$

The star now evolves along a sequence of states in which it is not only in hydrostatic but also in thermal equilibrium.

↓

complete (mechanical and thermal) *equilibrium*.

The ODEs of stellar evolution: timescales & simplifications

-In *complete equilibrium* equations split into 2 parts:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} , \quad \frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} , \quad \frac{\partial l}{\partial m} = \varepsilon_\mathrm{n} - \varepsilon_\nu , \quad \frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla ,$$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) , \quad i = 1, \ldots, I .$$

(1) the structure equations contain only spatial derivatives.

(2) The chemical equations contain only time derivatives.

The ODEs of stellar evolution: timescales & simplifications

-In *complete equilibrium* equations split into 2 parts:

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi r^2 \varrho} , \quad \frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} , \quad \frac{\mathrm{d}l}{\mathrm{d}m} = \varepsilon_\mathrm{n} - \varepsilon_\nu , \quad \frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{GmT}{4\pi r^4 P}\nabla ,$$

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \frac{m_i}{\varrho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) , \quad i = 1, \ldots, I .$$

(1) the structure equations contain only spatial derivatives.

(2) The chemical equations contain only time derivatives.

↓

with given $X_i(m,t_0)$ the structure equations are 4 ordinary differential equations (ODE).

The ODEs of stellar evolution: timescales & simplifications

-In *complete equilibrium* equations split into 2 parts:

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi r^2 \varrho} , \quad \frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} , \quad \frac{\mathrm{d}l}{\mathrm{d}m} = \varepsilon_\mathrm{n} - \varepsilon_\nu , \quad \frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{GmT}{4\pi r^4 P}\nabla$$

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \frac{m_i}{\varrho}\left(\sum_j r_j\right.$$

- Complete equilibrium is good *approximation* for stars in many evolutionary phases.
- The two parts of equations (d/dm, d/dt) are solved in two different, alternating steps with different numerical schemes.
- This introduces a basic inconsistency: 4 ODEs are solved at time, e.g., t_0 with given $X_i(m,t_0)$. After having solved $r(m)$,…,$T(m)$ some layers may have become convective with $X_i(m,t_0)$ having changed, and solution may no longer be consistent with real chemical composition. Mixing is done only in the next (time-)step, after the spatial problem is solved. The spatial solution in the next time step Δt is, however, still inconsistent, etc. Moreover, constancy of variable during time step: this inconsistency leads to an overestimation of main-sequence lifetimes. This inconsistencies can be kept small with sufficient small Δt.