Absorption cross section of microscopic black holes and their consequences for Earth

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Preface

This scientific paper will focus on investigating the topic of microscopic black holes from different perspectives and ultimately use it to asses their consequences to Earth. The purpose of the paper is to illuminate and convey interesting theory along the way, still with applicative uses to the overarching objective. As such, the reader is assumed to be familiar with basic concepts of general relativity and the notation thereof, as well as quantum theory.

The derivations and notation is heavily inspired by Sean M. Carroll in his book *Spacetime and Geometry: An Introduction to General Relativity*[1]. In preparation for the project I learned general relativity mostly by studying the book and am therefore inclined to apply the methods thereof.

Abstract

A lawsuit was filed against CERN due to the suspicion of the LHC potentially creating microscopic black holes that can endanger all of Earth. In this paper we investigate this claim through a semiclassical treatment of these singularities. First we will determine the classical absorption cross section of a Schwarzschild black hole. This result lets us ascertain how long it takes a microscopic black hole within the Earth to absorb an electron. Then we will find the quantum mechanical absorption cross section of the black hole, to determine if the result changes. This yields timescales of around 10^{55} years, which is beyond any relevancy for the lifespan of the Earth.

We will then investigate the properties of charged and rotating black holes, to find that they give rise to naked singularities, whose existence is questionable by the weak cosmic censorship hypothesis. From this along with Hawking radiation and rational argumentation we are lead to the final and overwhelming conclusion that use of the LHC and the potential creation of microscopic black holes is completely harmless.

Resumé

CERN blev sagsøgt under mistanken af at LHC potentielt kunne skabe mikroskopiske sorte huller, som kunne bringe hele Jorden i fare. I dette papir vil vi undersøge denne påstand gennem semiklassiske beregninger af disse singulariteter. Først vil vi bestemme det klassiske absorptionstværsnit af Schwarzschild sorte huller. Med dette resultat kan vi bestemme tiden, det tager et sort hul i Jorden, for at absorbere én elektron. Derefter vil vi bestemme det kvantemekaniske absorptionstværsnit for at se, om resultatet ændrer sig. Dette giver tidsskalaer på omkring 10^{55} år, hvilket er langt udover nogen relevans for Jordens levetid.

Vi vil derefter undersøge egenskaberne ved ladede og roterende sorte huller og finde, at de giver anledning til nøgne singulariter, hvis ekstistens er tvivlsom pr. *weak cosmic censorship hypothesis*. Alt dette sammen med Hawking stråling og rationel argumentation leder os til den endelige konklusion, at brugen af LHC og den potentielle dannelse af mikroskopiske sorte huller er total ufarlig.

Contents

1 Introduction

In March 2008, a retired nuclear safety officer, Walter Wagner, and a journalist, Luis Sancho, filed a lawsuit in order to prevent the operation of the Large Hadron Collider (LHC) in CERN [6]. Their concern was for the well being of the whole world, as they believed that the LHC would be able to create microscopic black holes that could have the potential to devour the Earth. The case was dismissed in the US on the grounds of the USA not having jurisdiction over the operation, but the judge also ruled that no sufficient evidence for their claim was given.

In this paper we will look further into this dilemma and ascertain whether the possible creation of microscopic black holes is any cause for concern in the first place. To this end, we will investigate the theoretical properties of black holes holes mainly concerning the absorption cross section. This entails the use of general relativity, where we will adhere to the sign convention $(-, +, +, +)$ throughout the paper. As for the units we will in all but chapter 3 and 4 set $c = 1$, as it helps visualise the scale of relativistic velocities and allows us to treat time and space equally.

We will initially calculate the classical absorption cross section of a Schwarzschild black hole. This will be used to give a semiclassical estimate of the time it takes a microscopic black hole oscillating within the earth to absorb an electron. Then we will derive a quantum mechanical approximation of the non-relativistic absorption cross section for microscopic black holes.

After the case of the Schwarzschild black hole we will look at the metrics concerning charged or rotating black holes and the properties thereof. Using all this along with Hawking radiation and rational argumentation we may give a qualified answer to the dilemma at hand.

2 Classical absorption cross section

In this chapter we will derive expressions for the absorption cross sections of a black hole. The derivation is based upon what is known as the Schwarzschild metric. This metric can be found as the unique solution to Einstein's equation in vacuum for a spherically symmetric and static source, and is given by[1]:

$$ds^2 = -d\tau^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 \tag{2.1}$$

Here $d\Omega^2$ is the metric of the unit two-sphere:

$$d\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2 \tag{2.2}$$

Notice how the metric does not contain any cross terms, thus only giving non-zero terms when the indices are identical. Rewriting the metric in terms of what is known as Eddington-Finkelstein coordinates one can show that the light-cone will warp in such a way that no future paths can move away from the black hole at $r = 2GM$ which is known as the Schwarzschild radius, denoted by R_s[1].

To study the absorption cross section we will look at how test particles of various speeds move in the space-time described by the metric. These free particles will move along geodesics, and therefore our initial goal will be to describe the radial equation of motion of the test-particle on the geodesic. Thus we can find the exact geodesics from which the particle will not escape.

2.1 Radial equation of motion

An initial thought would be to apply the geodesic equation, where one from symmetry arguments of energy and angular momentum can extract an equation purely dependent on the radial distance. We will however apply another method leaning on the fact that the following expression is constant[1]:

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{2.3}$$

where ϵ is a constant, and λ denotes some affine parameter. This can be seen as for the case of light, the particle will travel along a null path, so we have $\epsilon = 0$ by the definition of the metric. For massive particles we may divide by $-d\tau^2$ on both sides in the expression of the metric (2.1) which yields:

$$1 = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{2.4}$$

This is equivalent to eq. (2.3) if the affine parameter is chosen to be proper time, $\lambda = \tau$. This means eq. (2.3) will be valid for both massive and massless particles, with $\epsilon = 1$ and $\epsilon = 0$ respectively (remembering to use proper time for the massive particle). Now, observe that the metric has zero derivative in the time- and azimuthal direction. Using either Killing vectors or the Lagrangian along with spherical symmetry, one can derive the expressions for the conservation of energy and angular momentum along the geodesics implied by Noethers theorem. Thereby we arrive at the final expression[1]:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 \tag{2.5}$$

Where $V(r)$ denotes the effective potential describing the movement of the particle given by:

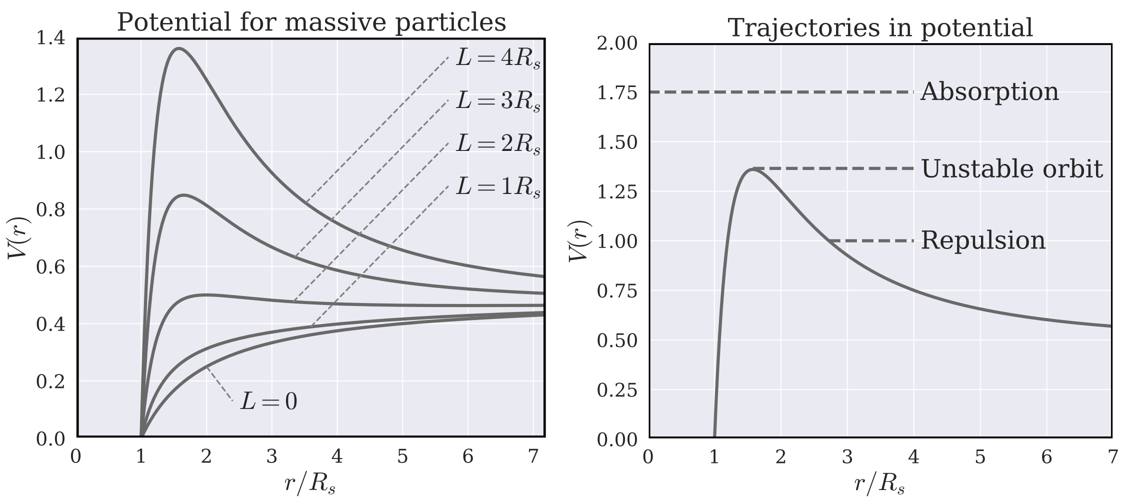
$$V(r) = \frac{1}{2}\epsilon - \frac{R_s}{2r}\epsilon + \frac{L^2}{2r^2} - \frac{R_s L^2}{2r^3} \tag{2.6}$$

The quantities E and L refer to energy and angular momentum for massless particles, and energy and angular momentum per unit mass for massive particles. Now a particle moving along a geodesic described by eq. (2.5) will move exactly as a particle in the potential $V(r)$ with total energy $\frac{1}{2}E^2$. This is evident as the equation is completely analogous to the usual equation for the sum of kinetic and potential energy.

In order to determine the absorption cross section of a black hole we assume some particle to approach the black hole from an infinite distance with speed v_∞ and impact parameter b. We want to determine the maximum impact parameter such that the particle is absorbed denoted by b_{max}. With this quantity the absorption cross section is simply given by:

$$\sigma_{abs} = \pi b_{max}^2 \tag{2.7}$$

Now if the effective energy, $\frac{1}{2}E^2$, is higher than the peak of the effective potential, $V(r)$, the particle will be absorbed, by the black hole. If it is lower, the particle will be repelled by the potential. At the exact border between the cases there is an unstable orbit [3].



(a) (b)

Figure 1:
(a) The potential for different values of angular momentum, L. Here one sees that the effective potential increases for larger values of L corresponding to larger impact parameters.
(b) The figure depicts the possible trajectories within the potential. These are absorption and repulsion, where the boundary yields an unstable orbit, which is the point of interest.

The impact parameter is proportional to angular momentum. Thereby the maximal absorption impact parameter for a given E, is found when the L is as large as possible, which exactly corresponds to when the effective energy is equal to the maximal value of the effective potential (see fig. (1)). This happens at a distance denoted by r_c, and can be described by the following equations:

$$V(r_c) = \frac{1}{2}E^2 \implies (E^2 - \epsilon)r_c^3 + \epsilon R_s r_c^2 - L^2 r_c + R_s L^2 = 0 \tag{2.8}$$

$$V'(r_c) = 0 \implies R_s r_c^2 \epsilon - 2L^2 r_c + 3R_s L^2 = 0 \tag{2.9}$$

The quantities E and L are constant along the geodesic, and therefore they may be specified at an infinite distance, where space-time is flat:

$$\text{massless particles:} \quad E = p \quad \text{and} \quad L = bp = bE \tag{2.10}$$

$$\text{massive particles:} \quad E = \frac{1}{\sqrt{1 - v_\infty^2}} \quad \text{and} \quad L = \frac{bv_\infty}{\sqrt{1 - v_\infty^2}} = bv_\infty E \tag{2.11}$$

Using these equation we may solve for the absorption cross section analytically in the limiting cases.

2.2 Cross section for massless and ultra-relativistic particles

For massless particles we have $\epsilon = 0$, which reduces eq. (2.9) to:

$$r_c = \frac{3}{2}R_s \tag{2.12}$$

This expression may be substituted into eq. (2.8) which yields:

$$E^2\frac{3^3}{2^3}R_s^3 - L^2\frac{3}{2}R_s + L^2R_s = 0 \implies b^2 = \frac{L^2}{E^2} = \frac{27}{4}R_s^2 \tag{2.13}$$

Eq. (2.8) and (2.9) ensures that we have the value of L that maximises the impact parameter given E. From this we can find the absorption cross section:

$$\sigma_0 = \pi b_{max}^2 = \frac{27\pi}{4}R_s^2 \tag{2.14}$$

Thereby this is the absorption cross section of a black hole for massless particles.

For particles in the ultra-relativistic regime E and L will diverge while $v_\infty \to 1$. From eq. (2.11) we know that $b = L/E$ in this limit. As we are working with massive particles, we may set $\epsilon = 1$. We can in eq. (2.8) and (2.9) ignore anything but the highest order in E and L as they diverge, but this reduces the equations to the exact equations for the case of massless particles, which we have already solved. As the impact parameter in this limit is also the same, the computations are exactly analogous to above, which yields:

$$\sigma_{UR} = \pi b_{max}^2 = \frac{27\pi}{4}R_s^2 \tag{2.15}$$

This result is to be expected as for large energies, the contribution of mass to the total energy becomes negligible, which results in an analogous case to that of massless particles.

2.3 Cross section for non-relativistic particles

For non-relativistic particles, we have $v_\infty \ll 1$, $E \approx 1$ [3] and $\epsilon = 1$. The motivation for $E \approx 1$ is that eq. (2.11) reduces to $b = L/v_\infty$, which is the classical formula for angular momentum. Rewriting eq. (2.8) we find:

$$R_s r_c^2 - L^2 r_c + R_s L^2 = 0 \tag{2.16}$$

Now the maximal impact parameter for absorption is found, when the L is as large as possible while still allowing for absorption. This is exactly the point where the horizontal line corresponding to an effective energy of 1/2 is barely touched by the graph of the potential, as any higher values of L would correspond to repulsion, see fig. (1). This is the case where the above equation only has one solution, so we may find the maximal L by setting the discriminant equal to zero:

$$L^4 - 4R_s^2 L^2 = 0 \implies L = 2R_s \implies b_{max} = 2\frac{R_s}{v_\infty} \tag{2.17}$$

Thereby the absorption cross section for non-relativistic particles is given by:

$$\sigma_{NR} = \pi b_{max}^2 = 4\pi \frac{R_s^2}{v_\infty^2} \tag{2.18}$$

Intuitively it would make sense for the cross section to diverge as the speed goes to zero as an almost stationary particle will be absorbed, even for very large impact parameters. This concludes the regime of analyticity from where we will solve for the absorption cross section numerically for any given speed.

2.4 Numerical calculation of cross section

To solve for the cross section numerically one may use sympy to solve a system of two equations consisting of eq. (2.8), (2.9) for r_c and b_{max}^2 with E and L defined by eq. (2.11). This yields the following:

$$b_{max}^2 = \frac{R_s^2}{8v_\infty^4}\left(8v_\infty^4 + 20v_\infty^2 + av_\infty^2(v_\infty^2 + 1/8)^{1/2} + b(v_\infty^2 + 1/8)^{1/2} - 1\right) \tag{2.19}$$

$$\text{where:} \quad a = 16\sqrt{2}, \quad b = 2\sqrt{2} \tag{2.20}$$

The code gave a and b numerically with a precision of up to 15 decimals, but these can be precisely identified with above exact expressions matching for all the stated decimals. Using eq. (2.7), inserting the constants, and simplifying then yields:

$$\sigma_{abs} = \frac{\pi R_s^2}{8v_\infty^4}\left(8v_\infty^4 + 20v_\infty^2 + (8v_\infty^2+1)^{3/2} - 1\right) \tag{2.21}$$

In order to check the legitimacy of this general solution, we can in the limits compare it to the analytically obtained solutions. For the ultra-relativistic limit we find:

$$\sigma_{abs}(v_\infty = 1) = \frac{27\pi}{4}R_s^2 = \sigma_{UR}, \tag{2.22}$$

as anticipated. For the non-relativistic limit ($v_\infty \ll 1$) we may first taylor expand the following expression:

$$(8v_\infty^2+1)^{3/2} = 1 + 12v_\infty^2 + O(v_\infty^3) \tag{2.23}$$

Now we may ignore anything but the lowest order of v_∞ inside the parentheses of eq. (2.21). The constant terms cancel leaving second order to be the lowest. Thus we find:

$$\sigma_{abs}(v_\infty \ll 1) \approx \frac{\pi R_s^2}{8v_\infty^4}\left(20v_\infty^2 + 12v_\infty^2\right) = 4\pi\frac{R_s^2}{v_\infty^2} = \sigma_{NR} \tag{2.24}$$

So we have found, that the numerical solution fits the analytical solutions in the limits, and thereby we have found an expression for the classical absorption cross section of black holes for any given initial speed.

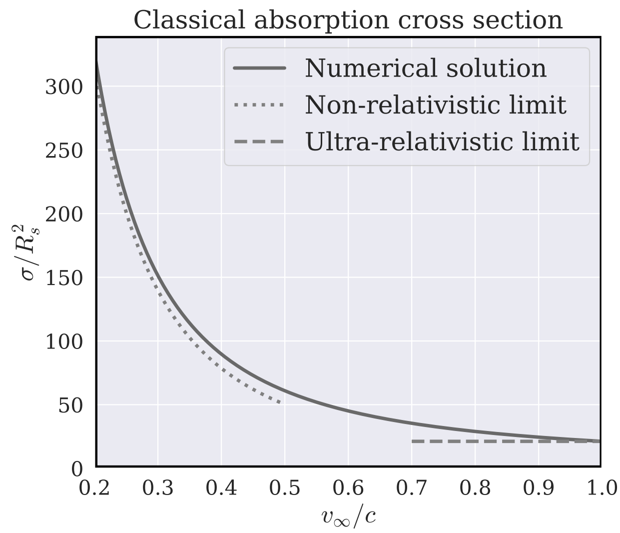


Figure 2: The figure depicts the classical absorption cross section of a Schwarzschild black hole as a function of the incident speed at an infinite distance. The analytic limiting cases are shown to be in accordance with the trend of the total numerical solution.

With this we may now give a semi-classical estimate of the time it would take a black hole to absorb a singular electron within the Earth.

3 Black hole moving through Earth

In this chapter we will explore what happens if a neutron were to be converted into a black hole, which would then pass through the earth. Would it absorb anything? We will work under the assumption that the neutron black hole is dropped from the surface of the earth and oscillates in the gravitational potential within the earth. To calculate if the black hole would absorb anything along its path we need the mean free path, which is given by:

$$l = \frac{1}{n\sigma}. \tag{3.1}$$

Here σ is the cross section of the black hole, which we determined earlier and n is the number density of particles in the earth.

3.1 Number density, potential, and RMS speed

Let us assume, that the earth has mass M and radius R. Electrons occupy most of the space within the earth, so we would like to approximate the number density of the electrons. To find the number of electrons we approximate, that 50% of the Earth's mass consists of protons, while the rest would be neutrons. Assuming the number of protons and electrons to be the same, the number of electrons would be half of Earths mass divided by the mass of a proton denoted m_p. Then we may divide by the volume of Earth to determine the number density. This is a rather crude approximation, but it will suffice as we only want suggestive estimates. This yields a number density of:

$$n = \frac{N}{V} = \frac{M}{2m_pV} = \frac{3M}{8\pi R^3 m_p} \approx 1.65 \times 10^{30}\,\frac{\text{electrons}}{\text{m}^3} \tag{3.2}$$

This value turns out to be almost exactly the same as if we estimated the earth to consist of electrons with each electron inhabiting their own sphere of Bohr radius, a_0, yielding $n = 3/(4\pi a_0^3) \approx 1.61 \times 10^{30}\,\frac{\text{electrons}}{\text{m}^3}$.

For the cross section, we will use the average kinetic energy to determine the root-mean-square speed of the black hole, as it moves through the earth. We then estimate the black hole to be constantly moving at this speed. First off the total energy may be written as the sum of the average kinetic energy and average potential energy:

$$E = \langle T \rangle + \langle U \rangle \tag{3.3}$$

The total energy E is just the potential energy at the dropping point $U(R)$. Thereby, we need an expression for $U(r)$. To do this we will assume the earth to have a homogeneous distribution of mass. Suppose the black hole has mass m and is at a distance r from the centre of the Earth. By Gauss's law of gravity, we may disregard all mass outside the radius r and treat everything within the radius as a point mass. Let $\tilde{M}$ denote this mass, which is given the ratio of volumes times the total mass, $\tilde{M} = Mr^3/R^3$. Then the force on the black hole is given by:

$$\mathbf{F}(r) = -\frac{Gm\tilde{M}}{r^2}\hat{r} = -\frac{GmM}{R^3}r\hat{r} \tag{3.4}$$

From this we may find the potential energy by integrating over the work. We set the zero point of the potential to be in the centre of the earth:

$$U(r) = -\int_0^r (\mathbf{F}(r') \cdot \hat{r})\, dr' = \frac{GmM}{R^3}\int_0^r r' dr' = \frac{GmMr^2}{2R^3} \tag{3.5}$$

Now we may determine the average kinetic energy in terms of the average potential energy using the virial theorem:

$$\langle T \rangle = \frac{1}{2}\left\langle r\frac{dU}{dr}\right\rangle = \frac{1}{2}\left\langle r\frac{GmMr}{R^3}\right\rangle = \langle U \rangle \tag{3.6}$$

This result is expected, as the system is a harmonic oscillator as seen by the expression of the force. We may now insert this into the equation for the total energy setting $E = U(R)$ and solve for the root-mean-square speed:

$$U(R) = 2\langle T \rangle = mv_{rms}^2 \tag{3.7}$$

$$\Longrightarrow v_{rms} = \sqrt{\frac{U(R)}{m}} = \sqrt{\frac{GM}{2R}} \tag{3.8}$$

This can be calculated to be $v_{rms} = 5593\frac{\mathrm{m}}{\mathrm{s}}$ or 1.866×10^{-5} in units of c.

3.2 Mean free path and period for absorption

With this approximation we may now determine the mean free path of the black hole:

$$l = \frac{1}{n\sigma(v_{rms})} = \frac{8\pi R^3 m_p}{3M\sigma(v_{rms})} \approx 2.7 \times 10^{66}\mathrm{m} \tag{3.9}$$

From this we may approximate the time for the black hole to absorb an electron by:

$$t = \frac{l}{v_{rms}} \approx 1.5 \times 10^{55} yr \tag{3.10}$$

So if the black hole were to oscillate within the earth it would still take 10^{55} years before it absorbs an electron which is way beyond any measurable time in our universe. Of course many of the approximations were crude, but the general timescale is so large, that even large changes to the result would have no impact on the conclusion. Thus, the time for absorption is way to long to carry any relevance or concern on our part.

4 Quantum mechanical absorption cross section

The absorption cross section, we derived, is purely classical. As we are working with microscopic black holes, it begs us to consider the problem using quantum mechanics. In the last chapter we found $v_{rms} = 5593\frac{\mathrm{m}}{\mathrm{s}}$, so we will work in the non-relativistic regime. This allows us to apply the Schrödinger equation to calculate the absorption cross section. We will in this chapter follow the derivations of L. Z. Fang and R. C. Wang in their article, *Absorption cross sections of a Schwarzschild black hole*, 1983 [2]. The article is very brief omitting almost all of the calculations and discussion. Therefore, we will expand upon the derivations, to see how the results emerge from the theory and the implications thereof.

4.1 Derivation of effective Schrödinger equation

The idea of the derivation is to write the Klein-Gordon equation in curved space-time. This is necessary as we need to start from an equation that treats time and space at an equal footing, such that it is compatible with relativistic theory. Then by evaluating this in the non-relativistic limit, we recover the Schrödinger equation from where we can identify the effective potential. This potential may then be used to determine the absorption cross section. Throughout the derivation all quantum states are implicitly taken to be in position space, $\psi = \langle \mathbf{x}|\psi\rangle$. To do the calculations, we need the Schwarzschild metric in a coordinate system, that is not singular at the event horizon. Therefore, we define a new time coordinate, t', by:

$$ct' = ct + (r^* - r), \quad \text{where} \quad r^* = r + R_s \ln\left(\frac{r}{R_s} - 1\right) \implies \frac{dr^*}{dr} = \left(1 - \frac{R_s}{r}\right)^{-1}. \tag{4.1}$$

Here r^* is called the Tortoise coordinate. Changing time to the t' coordinate yields the metric in Eddington-Finkelstein coordinates:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) c^2 \mathrm{d}t'^2 + \left(1 + \frac{R_s}{r}\right) \mathrm{d}r^2 + \frac{R_s c}{r}(\mathrm{d}t'\mathrm{d}r + \mathrm{d}r\mathrm{d}t') + r^2 d\Omega^2 \tag{4.2}$$

From now on we will for simplicity denote the time coordinate, t', by t, keeping the new definition in mind. Note that the spacial coordinates are still the usual spherical coordinates. To determine $g^{\mu\nu}$ one may simply use the $g_{\mu\lambda}g^{\lambda\nu} = \delta^{\nu}_{\mu}$. Now we are ready to state the Klein-Gordon equation in curved spacetime:

$$\hbar^2 g^{\mu\nu}\nabla_\mu\nabla_\nu\Psi = m^2c^2\Psi, \tag{4.3}$$

where ∇_μ denotes the covariant derivative defined on covectors by:

$$\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^\lambda{}_{\mu\nu} w_\lambda, \quad \text{where} \quad \Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{4.4}$$

$\Gamma^\lambda{}_{\mu\nu}$ are known as the Christoffel symbols. With this we may rewrite the Klein-Gordon equation. Note that the covariant derivative of a scalar function is just the regular partial derivative, as the function is independent of choice of coordinates. Thereby we may write:

$$\hbar^2 g^{\mu\nu}\nabla_\mu\nabla_\nu\Psi = \hbar^2 g^{\mu\nu}\nabla_\mu\partial_\nu\Psi = \hbar^2 g^{\mu\nu}(\partial_\mu\partial_\nu\Psi - \Gamma^\lambda{}_{\mu\nu}\partial_\lambda\Psi) = m^2c^2\Psi \tag{4.5}$$

Now we may use the metric to write out the Klein-Gordon equation. The computations are simple but extremely tedious. The Christoffel symbols are symmetric in the lower indices, and we only need the ones where the lower indices correspond to nonzero metric entries by above equation. This still leaves 20 symbols to be calculated. We will omit these calculations, but one may use, that the metric only has cross terms in r and t, and the components only depend on r and θ to simplify the computations. This yields the following expression:

$$\begin{aligned}\hbar^2\left(-\left(1+\frac{R_s}{r}\right)\frac{1}{c^2}\partial_t^2+\frac{2R_s}{cr}\partial_r\partial_t+\frac{R_s}{cr^2}\partial_t-\frac{R_s}{r^2}\partial_r r\partial_r\right)\Psi\\+\hbar^2\left(\frac{1}{r^2}\partial_r r^2\partial_r+\frac{1}{r^2\sin(\theta)}\partial_\theta\sin(\theta)\partial_\theta+\frac{1}{r^2\sin^2(\theta)}\partial_\phi^2\right)\Psi=m^2c^2\Psi\end{aligned} \tag{4.6}$$

The second term can be recognised as the Laplacian in spherical coordinates. Now we have our final expression, from where we may compute it in the non-relativistic limit. Let us assume $\Psi = exp(-i\varepsilon t/\hbar)\psi$ to be a stationary solution. Inserting the stationary solution above allows us to substitute $\partial_t \to -i\varepsilon/\hbar$, from where we may divide out the time dependent part of the solution. In the non-relativistic limit we may write $\varepsilon = mc^2 + E$, where E is the sum of potential and kinetic energy. $E \ll mc^2$, so we will approximate ε^2 to be in first order in E, $\varepsilon^2 \approx m^2c^4 + 2mc^2E$. This yields:

$$2m\left(E-V_N\left(1+2\frac{E}{mc^2}\right)-i\left(1+\frac{E}{mc^2}\right)V_2-V_1\right)\psi+\hbar^2\nabla^2\psi=0 \tag{4.7}$$

$$\text{where}\quad V_N=-\frac{GMm}{r},\quad V_1=\frac{\hbar^2}{2m}\frac{R_s}{r^2}\partial_r r\partial_r,\quad V_2=R_s\hbar c(\frac{1}{r}\partial_r+\frac{1}{2r^2}) \tag{4.8}$$

Now setting the terms with $E/mc^2 = 0$ as they are negligible compared to 1 and rewriting finally yields:

$$-\frac{\hbar^2}{2m}\nabla^2\psi+(V_N+V_1+iV_2)\psi=E\psi \tag{4.9}$$

4.2 Calculation of the absorption cross section

Now we have the Schrödinger equation, and we may identify the effective potential as $V_N + V_1 + iV_2$. A complex potential allows for absorption of particles which is exactly what we desire. We will regard $V_1 + iV_2$ as perturbations to the classical gravitational potential V_N. Note that solutions to the unperturbed Schrödinger equation are equivalent to the solutions for the Coulomb potential with $ze^2/4\pi\epsilon_0 \to GMm$. Thus, we will analogously denote the unperturbed solutions as $\psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \phi)$.

To determine the absorption cross section, we will derive an expression for the change in probability density within the black hole for the total potential. Then we will use the unperturbed scattering state to determine the first order approximation of the cross section. Suppose Ψ is a solution of the time dependent Schrödinger equation of the total potential.

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_N\Psi + V_1\Psi + iV_2\Psi \tag{4.10}$$

By taking the complex conjugate of both sides, one finds the equation for Ψ^*:

$$-i\hbar\partial_t\Psi^* = -\frac{\hbar^2}{2m}\nabla^2\Psi^* + V_N\Psi^* + V_1\Psi^* - iV_2\Psi^* \tag{4.11}$$

This follows as each operator commutes with complex conjugation. From this we can find the derivative of the probability density, $\rho = |\Psi|^2$:

$$\begin{aligned}\partial_t\rho =& \frac{-i}{\hbar}i\hbar\partial_t(\Psi\Psi^*) = \frac{-i}{\hbar}((i\hbar\partial_t\Psi)\Psi^* - \Psi(-i\hbar\partial_t\Psi^*)) \\ =& \frac{-i}{\hbar}\left(-\frac{\hbar^2}{2m}(\nabla^2\Psi)\Psi^* + \frac{\hbar^2}{2m}\Psi(\nabla^2\Psi^*) + V_N\Psi\Psi^* - \Psi V_N\Psi^*\right) \\ &- \frac{i}{\hbar}((V_1\Psi)\Psi^* - \Psi V_1\Psi^* + i(V_2\Psi)\Psi^* + i\Psi V_2\Psi^*)\end{aligned} \tag{4.12}$$

Notice that the terms concerning V_N cancel, as V_N does not contain derivatives and thereby commutes with Ψ. Also note that Ψ^* commutes with $V_1\Psi$. From this we may write:

$$
\begin{aligned}
\partial_t \rho =& -\nabla \cdot \left(\frac{i\hbar}{2m} (\Psi(\nabla\Psi^*) - (\nabla\Psi)\Psi^*) \right) \\
&+ \frac{1}{\hbar}(i(\Psi V_1 \Psi^* - \Psi^* V_1 \Psi) + \Psi V_2 \Psi^* + \Psi^* V_2 \Psi) \\
=& -\nabla \cdot \left(\frac{-\hbar}{m} \operatorname{Im}\{\Psi \nabla \Psi^*\} \right) + \frac{2}{\hbar}(-\operatorname{Im}\{\Psi V_1 \Psi^*\} + \operatorname{Re}\{\Psi V_2 \Psi^*\}) \\
=& -\nabla \cdot \mathbf{j} + \frac{2}{\hbar}(-A + B)
\end{aligned} \tag{4.13}
$$

where $\mathbf{j}$ denotes the probability flux. This expression differs from the one in the article, where they did not have the factor of i in front of the term concerning V_1. This factor is however necessary in order for the term to be real, which is a condition for the change in probability density. This implies some mistake of the equation in the article. This inconsistency does through later derivations prove to be inconsequential to further results. Another inconsistency is that the sign in front of $(-A+B)$ is negative in the article as opposed to the positive sign here. This will result in a change of the total sign of the cross section, as seen later.

Now to find the change in probability density to first order we will insert the unperturbed scattering solution in the right hand side of the equation above. Before doing so, note that solutions to the unperturbed Schrödinger equation $R_{nl}(r)Y_l^m(\theta, \phi)$ has a radial function exclusively in the real numbers. As the operator V_1 only acts on the radial part, $\Psi V_1 \Psi^*$ is real, thus having zero imaginary part implying $A = 0$. The unperturbed scattering state also has zero divergence in probability flux [9] by conservation of probability, which means that an unperturbed solution, ψ_0 must yield the following:

$$
\partial_t \rho^{(1)} = \frac{2}{\hbar} B = \frac{1}{\hbar}(\psi_0 V_2 \psi_0^* + \psi_0^* V_2 \psi_0)
$$

With $\partial_t \rho^{(1)}$ denoting the first order approximation. Let us denote the space within the event horizon as U, and the outside of the black hole as the spacial complement U^C. The absorption cross section is given by the total absorption per unit time, divided by the velocity of an incoming free particle wave[2] which is $k\hbar/m$. One motivation for this is that the velocity is proportional to the probability flux of a free wave, which matches the definition of the cross section as absorbed probability divided by incoming probability flux. Notice this yields the desired units of area for the cross section:

$$\sigma_{QM}^{(1)} = \frac{\int_U \partial_t \rho^{(1)} d^3\mathbf{r}}{v} = \frac{-m}{k\hbar}\int_{U^C} \partial_t \rho^{(1)} d^3\mathbf{r} = \frac{-m}{k\hbar^2}\int_{U^C} (\psi_0 V_2 \psi_0^* + \psi_0^* V_2 \psi_0) d^3\mathbf{r}, \quad (4.14)$$

We used that the integral of $\partial_t \rho$ over U is equal and opposite to the integral over U^C due to conservation of total probability. This is because we view absorption as an increase in total probability within the black hole, which likewise decrease the total probability outside of the event horizon. This expression is equivalent to the one stated in the article with the exception of an opposite sign in front of the integral. This was anticipated by the inconsistency in signs between the two expressions of probability density. We will nevertheless use the expression we derived, which will later turn out to yield the desired answer.

To calculate the absorption cross section we will insert the unperturbed scattering state, ψ_0, written in partial fraction expansion. Coulomb scattering is a well-discussed topic, so we will just state the expression of the state [2]:

$$\psi_0 = \sum_{l=0}^{\infty} C_l r^l e^{ikr} F(l+1-i; 2(l+1); -2ikr) P_l(\cos(\theta)) = \sum_{l=0}^{\infty} f_l(r) P_l(\cos(\theta))$$
$$\text{where} \quad C_l = (2ikr)^l e^{\alpha\pi/2}\Gamma(l+1-i\alpha)/2l! \quad (4.15)$$

Here $k = 2mE/\hbar^2$ and $\alpha = Gm^2M/\hbar^2 k$. Γ is the gamma function, P_l is the l'th legendre polynomial and F is the confluent hypergeometric function[1]. Let us denote

[1]See Sakurai and Napolitano [9] pp. 203 for a definition and treatment with the coulomb potential.

the r dependent part of the sum as f_l for simplicity. With this the approximate absorption cross section is given by:

$$\sigma_{QM}^{(1)} = -\frac{m}{k\hbar^2}\int_0^{2\pi} d\phi \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\pi} P_l(\cos(\theta))P_m(\cos(\theta))\sin(\theta)d\theta$$
$$\times \int_{R_s}^{\infty} r^2(f_l(r)V_2 f_m^*(r) + f_m^*(r)V_2 f_l(r))dr \qquad (4.16)$$

Here we used, that V_2 only acts on r and that the Legendre polynomials are real along with $d^3\mathbf{r} = r^2\sin(\theta)d\phi d\theta dr$. The integral over ϕ trivial and equal to 2π. To calculate the integral over θ we may use the substitution $u = \cos(\theta)$ which yields:

$$\int_0^{\pi} P_l(\cos(\theta))P_m(\cos(\theta))\sin(\theta)d\theta = \int_{-1}^{1} P_l(u)P_m(u)du = \frac{2}{2l+1}\delta_{lm}. \qquad (4.17)$$

The last equality stems from a well-known identity of the Legendre polynomials [4]. The Kronecker delta, δ_{lm}, allows us to write the sum only in l, from where the θ-integral is $2/(2l+1)$. All that is left is the integral for r, setting $m = l$. Here we may insert the expression for V_2, eq. (4.8), from where we can recognise the product rule, to rewrite it as a single derivative:

$$\begin{aligned}
&R_s\hbar c\int_{R_s}^{\infty} r f_l(r)(\partial_r f_l^*(r)) + r(\partial_r f_l(r))f_l^*(r) + f_l(r)f_l^*(r)dr \\
=&R_s\hbar c\int_{R_s}^{\infty} \partial_r(r f_l(r)f_l^*(r))dr = -R_s\hbar c(R_s f_l(R_s)f_l^*(R_s)) \\
=&-R_s^2\hbar c\,|f_l(R_s)|^2 \qquad (4.18)
\end{aligned}$$

Here we used that f_l should be zero at infinity due to a finite total probability of the quantum state. Inserting all the integrals yields our approximate cross section:

$$\sigma_{QM}^{(1)} = \frac{4\pi m R_s^2 c}{k\hbar}\sum_{l=0}^{\infty}\frac{|f_l(R_s)|^2}{2l+1} \qquad (4.19)$$

Here we see, that the sign is correct, which implies, that our initial expression for the absorption cross section was correct as opposed to the one stated in the article.

We will now look at the limit of small black holes. In this limit $kR_s \ll 1$. From the expression of the unperturbed scattering state, eq. (4.15), one sees that $C_l \propto (kR_s)^l$. Thereby, we may ignore all terms excepts the contribution from the s-wave, as it dominates in this limit. This yields the final expression for the cross section[2]:

$$\sigma_{QM}^{(1)} \approx \frac{4\pi m R_s^2 c}{k\hbar} |f_0(R_s)|^2 = \frac{8\pi^2 R_s^2 m c\alpha}{\hbar k[1 - \exp(-2\pi\alpha)]} \tag{4.20}$$

Thereby, we have found an approximation of the quantum mechanical absorption cross section of microscopic black holes in the non-relativistic limit. We may compare this result with the clasiccal non-relativitsic result obtained in chapter 2, eq. (2.18). Note that the quantum cross section is not proportional to the inverse of k, as α also depends on k. For the case of a microscopic black hole with an electron, α is extremely small, $\propto 10^{-40}$, so expanding the exponential to the first order in α is justified. Then we may also rewrite using $v = k\hbar/m$ as to compare with the classical cross section. In this estimate, the cross section is given by:

$$\sigma_{QM}^{(1)} \approx \frac{8\pi^2 R_s^2 m c\alpha}{\hbar k 2\pi\alpha} = \frac{4\pi R_s^2}{(v/c)} \tag{4.21}$$

This result looks a lot like the classical result found in the non-relativisic regime eq. (2.18) (Note $c = 1$ in this equation). One would imagine the classical limit to be reached for such a small black hole as the angular momentum of the electron would have to be extremely small to compare. The peculiar thing is, that the quantum mechanical cross section is proportional to the inverse speed, while the classical cross section is proportional to the inverse squared. This may imply some fundamental error in the quantum mechanical calculations, which could be attributed to how quantum mechanics and general relativity are for now incompatible.

Another idea could be the way in which the absorption cross section is defined, or the way we made the first order approximation of the cross section. Nevertheless this is our result which is the same as in the article, and the more general result from eq. (4.19) is applicable and instructive for many other cases. If we propagate our new absorption cross section through the calculations of chapter 3 one finds the time for absorption of a single electron to be $t \approx 8 \times 10^{59}$ years, which is an even longer timescale. Thereby, the earlier conclusion from using the classical results still stands, but we have now verified it using quantum theory.

Now we have done a theoretical treatment of Schwarzschild black holes both using classical and quantum mechanics. One would imagine it possible that the LHC may also create charged or rotating black holes. If a proton were to become a black hole, there should be conservation of charge and angular momentum, carrying over as properties of the black hole. Therefore, we will in the next chapters discuss these cases and the implications thereof.

5 Charged black holes

So far all physical considerations have been made with respect to a Schwarzschild black hole, which entails neither charge nor spin. Therefore, we will in this chapter expand upon the theory by considering a charged black hole. The corresponding metric known as the Reissner–Nordström metric, is given by [7]:

$$ds^2 = -d\tau^2 = -\Delta \mathrm{d}t^2 + \Delta^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 \tag{5.1}$$

where,

$$\Delta = 1 - \frac{R_s}{r} + \frac{R_Q^2}{r^2} \quad \text{with} \quad R_s = 2MG \quad \text{and} \quad R_Q^2 = \frac{Q^2 G}{4\pi\epsilon_0}. \tag{5.2}$$

Here M is as usual the mass of the black hole, Q denotes the charge. $d\Omega^2$ is the metric of the unit two-sphere as before. The metric is a solution to Einsteins equation given by[1]:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5.3}$$

Here $G_{\mu\nu}$ is the Einstein tensor describing the curvature of spacetime, and $T_{\mu\nu}$

is the energy momentum tensor. Now this metric is no longer a vacuum solution to Einsteins equation as we have a nonzero energy-momentum density around the black hole due to the electromagnetic field. Inserting the metric into the equation allows us to express the electromagnetic potential, which turns out to be[7]:

$$A_\alpha = \left(-\frac{Q}{4\pi\epsilon_0 r}, 0, 0, 0\right) = \left(-\frac{\tilde{Q}}{r}, 0, 0, 0\right) = -\frac{\tilde{Q}}{r}\delta^t_\alpha \tag{5.4}$$

Where $\tilde{Q} = Q/4\pi\epsilon_0$. Now we will describe the motion of a test particle with charge q moving in this system. We assume the particle to have no influence on the gravitational and electromagnetic fields. This is a strong assumption for the case of interest, as the electron has equal and opposite charge compared to the proton black hole. Nevertheless, this greatly simplifies the calculations, so we will stick to this assumption. The test particle will no longer move along geodesics. Instead the motion of the particle is described through the Lorentz force.

5.1 Radial equation of motion

Now instead of employing a Newtonian approach we will follow the method of the paper *The Reissner-Nordström metric* by Jonatan Nordebo [7] which applies the Lagrange formalism to derive the radial equation of motion. First off, if we divide the expression for the metric, (5.1), by $d\tau^2$ and multiply by Δ we obtain the following expression:

$$\dot{r}^2 - \Delta^2\dot{t}^2 + \Delta + \Delta r^2\dot{\theta}^2 + \Delta r^2\sin^2(\theta)\dot{\phi}^2 = 0. \tag{5.5}$$

Here the dot represents the derivative with respect to proper time. We want an expression purely dependent on r. To achieve an expression for the conserved quantities we will as stated employ the Lagrange formalism. The Lagrangian of the system is given by [7]:

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + qA_\alpha\dot{x}^\alpha \tag{5.6}$$

Notice that the Lagrangian is independent of t and ϕ. Thereby, it is invariant under time translation and rotation which by Noether's theorem implies conservation of energy and the component of angular momentum along the z-axis. By spherical symmetry of the metric, the total angular momentum is then conserved as rotating the system yields conservation of the other components. Thus we may without loss of generality assume the particle to move in the plane normal to the z-axis defined by $\theta = \pi/2$. Inserting the expression for $g_{\mu\nu}$ and A_α, as well as setting $\theta = \pi/2$ yields:

$$\mathcal{L} = -\frac{1}{2}\Delta\dot{t}^2 + \frac{1}{2}\Delta^{-1}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{q\tilde{Q}}{r}\dot{t} \tag{5.7}$$

Now we can use the Euler-Lagrange equation given by:

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} = \frac{\partial\mathcal{L}}{\partial x^\alpha} \tag{5.8}$$

The derivatives with respect to t and ϕ are zero. By the above equation this implies the derivative with respect to $\dot{t}$ and $\dot{\phi}$ to be constant, thus yielding expressions for our conserved quantities. These are given by:

$$\Delta\dot{t} + \frac{q\tilde{Q}}{r} = E \tag{5.9}$$

$$r^2\dot{\phi} = L \tag{5.10}$$

Now we may substitute the expressions for E and L into our equation of motion, (5.5):

$$\dot{r}^2 - \left(E - \frac{q\tilde{Q}}{r}\right)^2 + \Delta\left(1 + \frac{L^2}{r^2}\right) = \dot{r}^2 + V(r) = 0. \tag{5.11}$$

We may again view this expression as analogous to the sum of kinetic energy and potential energy set equal to zero. This time we cannot separate E from r, so the potential will also have dependence on E. This means that the this is not an actual potential and does not obey the same rules as a physical potential. As an example $V(r \to \infty) = 1 - E^2$ which does not have to equal zero, which it must if the potential was physical. Nevertheless we will refer to it as a potential, as it still dictates the movement of the particle in accordance with above equation. If we set $Q = 0$ we recover eq (2.5), as expected, as the case should reduce to that of a Schwarzschild black hole.

5.2 Analysis of the potential and event horizon

In order to understand the movement of the particle, it would be instructive to analyse the properties of the potential. Writing it out and collecting the terms yields:

$$V(r) = (1 - E^2) - \frac{R_s - 2Eq\tilde{Q}}{r} + \frac{R_Q^2 + L^2 - q^2\tilde{Q}^2}{r^2} - \frac{R_s L^2}{r^3} + \frac{R_Q^2 L^2}{r^4} \tag{5.12}$$

The first thing to notice is that the coefficient of the last term is positive. As this term dominates for small values of r the potential will then diverge towards positive infinity. As the term does not contain q, this barrier will be present even for geodesic particle movement ($q = 0$). This can seem indicative, that the potential barrier never allows for absorption of the electron when it moves according to the Lorentz force or along geodesics. The saving grace could be if this barrier lies within the event horizon such that the electron will still be absorbed as no time-like path can return from beyond this horizon. Therefore it is of interest to calculate the possible positions of the event horizon.

The event horizon can be described as a null hypersurface[1]. This is a 3 dimensional hypersurface in our 4 dimensional space-time manifold that is generated by a collection of null geodesics. These are geodesics along which the proper time is zero. Thereby, the event horizon is a hypersurface, where all tangent vectors are null vectors. The necessity of this property follows from the definition that no timelike path can escape to infinity from beyond the event horizon. One may think of it as the boundary of the set of all timelike paths that can ever reach infinity. If the corresponding normal vectors of a hypersurface are null, then the hypersurface itself is null[1].

Often one needs a smart choice of coordinates to find such hypersurface. In our case, the metric is particularly elegant as it is stationary, spherically symmetric and asymptotically flat. As we are looking for spherical event horizons our current polar coordinates are sufficient. Consider the hypersurface described by $r = const$ with the corresponding normal vector $\partial_\mu r$ [1]. We can then determine the event horizon by setting the norm of the normal vector to zero:

$$0 = g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr} = (g_{rr})^{-1} = \Delta = 1 - \frac{R_s}{r} + \frac{R_Q^2}{r^2} \tag{5.13}$$

The expression of g^{rr} follows directly from $g_{\mu\lambda}g^{\lambda\nu} = \delta_\mu^\nu$ and the fact that the metric is diagonal. The position of the event horizon is the solution to above equation, and rewriting it yields:

$$r^2 - R_s r + R_Q^2 = 0 \tag{5.14}$$

$$\implies r_\pm = \frac{R_s \pm \sqrt{R_s^2 - 4R_Q^2}}{2} \tag{5.15}$$

From here we can see that the number of event horizons is determined by the discriminant. The point where the number of solutions changes is when $R_s = 2R_Q$. Calculating these quantities for the case of a proton yields in SI-units:

$$R_s = 2.484 \times 10^{-54}\ \mathrm{m} \quad \text{and} \quad R_Q = 1.381 \times 10^{-36}\ \mathrm{m} \tag{5.16}$$

Here R_s is much smaller than $2R_Q$, which means that the "proton black hole" is a naked singularity which is a singularity without an event horizon. Even under the assumption that such physical structures exist, the singularity will not absorb the electron, as motion in accordance with the Lorentz force will be subject to an infinite potential barrier. Without the event horizon, all particles will then be repulsed be it geodesic or Lorentz motion. This does not exclude the possibility for other timelike paths reaching the singularity, but this is not relevant for our case.

Now one may also argue whether the creation of naked singularities is even physically possible. Many believe in the Weak Cosmic Censorship Hypothesis formulated by Roger Penrose in 1969 [8]. It states that all singularities must be hidden from an observer at infinity via an event horizon, thus neglecting the existence of naked singularities.

6 Rotating black holes

As we have now considered charged black holes, we will address the case of rotating black holes. For generality, we will look at black holes, that are both charged and rotating. These are described using the Kerr-Newman metric which is given by[5]:

$$ds^2 = -d\tau^2 = \frac{\rho^2}{\Lambda}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 - \frac{\Lambda}{\rho^2}(a\sin^2(\theta)\mathrm{d}\phi - \mathrm{d}t)^2 + \frac{\sin^2(\theta)}{\rho^2}((r^2+a^2)\mathrm{d}\phi - a\mathrm{d}t)^2 \tag{6.1}$$

The coordinates (r, θ, ϕ) are no longer polar coordinates but Boyer–Lindquist coordinates, which relate to the usual spacial Cartesian coordinates by:

$$x = \sqrt{r^2+a^2}\sin(\theta)\cos(\phi), \quad y = \sqrt{r^2+a^2}\sin(\theta)\sin(\phi), \quad z = r\cos(\theta). \tag{6.2}$$

The quantities a, ρ^2 and Λ are given by:

$$a = \frac{J}{Mc}, \quad \rho^2 = r^2 + a^2\cos^2(\theta), \quad \Lambda = r^2 - R_s r + a^2 + R_Q^2, \tag{6.3}$$

where J is the angular momentum of the black hole. The corresponding electromagnetic potential is potential is:

$$A_\mu = \left(-\frac{rR_Q}{\rho^2}, 0, 0, \frac{arR_Q \sin^2(\theta)}{\rho^2}\right) \tag{6.4}$$

We see if $J = 0$ then $a = 0$, where our coordinates become the usual polar coordinates and the metric and potential become that of case of the Reissner–Nordström metric as expected.

When we previously derived the radial equation of motion, we always capitalised on the fact that the metric was spherically symmetric, and that we had a conserved current through independence of the variable ϕ. This allowed us to restrict the motion of the particle to a plane by setting $\theta = \pi/2$. For this case, we still have a conserved current in ϕ, but there is no spherical symmetry. Thereby, we cannot restrict ourselves to a singular plane, and the equations of motion are dependent on the orientation of the test particle with respect to the black hole.

Calculating these trajectories are outside of the scope of this paper. We may however as before analyse the possible positions of event horizons. Analogous to before, we will consider the hypersurface given by $r = const$, and set the norm of the normal vector $\partial_\mu r$ equal to zero:

$$0 = g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr} = (g_{rr})^{-1} = \frac{\Lambda}{\rho^2} \tag{6.5}$$

The expression for g^{rr} follows as there are no cross terms for r in the metric. $\rho^2 > 0$, everywhere but at the singularity, which reduces the equation to $\Lambda = 0$. Solving this yields:

$$r_\pm = \frac{R_s \pm \sqrt{R_s^2 - 4(a^2 + R_Q^2)}}{2}. \tag{6.6}$$

To find the number of event horizons, we have to check the sign of the expression within the square-root. Considering the proton singularity with $J = \frac{1}{2}\hbar$ we will in

SI-units find:

$$R_s = 2.484 \times 10^{-54}\,\mathrm{m}, \quad R_Q = 1.381 \times 10^{-36}\,\mathrm{m} \quad \text{and} \quad a = 1.0501 \times 10^{-16}\,\mathrm{m}. \tag{6.7}$$

From this we can see, that the discriminant is negative even if $R_Q = 0$. Thereby the singularities that are both rotating and charged or just rotating will not have any event horizons. Thus we are dealing with a naked singularities once again. This is the same situation as the charged black hole, and by the weak cosmic censorship hypothesis their existence is questionable.

7 Hawking Radiation

As we are treating microscopic black holes, it is of interest to consider the effect of Hawking radiation. This effect dictates that the black holes will evaporate over time, which only serves to strengthen our earlier results.

Hawking radiation is a consequence of the Unruh Effect which states that an observer moving with a uniform acceleration through the Minkowski vacuum will observe a thermal spectrum of particles given by the equation [1]:

$$T = \frac{a}{2\pi}, \tag{7.1}$$

where $a = \sqrt{a_\mu a^\mu}$ is the magnitude of acceleration. Suppose we are dealing with a metric giving rise to a static and asymptotically flat spacetime with an event horizon. As the metric is static we have the Killing vector given by $K^\mu = (\partial_t)^\mu = \delta^\mu_t$. For a static observer whose four-velocity is proportional to K^μ, the acceleration is given by [1]:

$$a_\mu = \nabla_\mu \ln(V), \quad \text{where} \quad K^\mu = VU^\mu. \tag{7.2}$$

V is called the redshift factor. For the Schwarzschild metric these quantities can be found to be [1]:

$$a = \frac{GM}{r^2\sqrt{1-\frac{2GM}{r}}}, \quad \text{and} \quad V = \sqrt{1-\frac{2GM}{r}} \tag{7.3}$$

The hawking radiation can be found by applying the Unruh effect to a black hole near an event horizon and using the redshift factor to account for an observer at infinity. For an asymptotically flat spacetime this yields [1]:

$$T_H = \lim_{r\to 2GM} \frac{Va}{2\pi} = \frac{\kappa}{2\pi} = \frac{1}{8\pi GM}, \tag{7.4}$$

Where $\kappa = 1/4GM$ is the surface gravity by the event horizon. Thereby a Schwarzschild black hole evaporates. This yields a lifetime of order $(M/M_\odot)^3 \times 10^{71}$ sec, which for the neutron black hole means a lifetime of order 10^{-101} sec. This only works for singularities with event horizons, so we cannot apply it to the charged or rotating black hole.

8 Discussion

So far we have uncovered a variety of topics concerning microscopic black holes. First we assumed the particles to be approximated by a Schwarzschild black hole. A semiclassical treatment showed that a microscopic black hole moving through Earth would not even absorb a single electron in any time span relevant for the duration of our universe. This was further reinforced using the quantum mechanical cross section. Through hawking radiation we found them to have an extremely short lifespan, which further discredits the idea of microscopic black holes being dangerous.

Then we showed that classically, microscopic charged black holes will repel all test particles moving in accordance with both geodesic and Lorentz motion. For both rotating and charged black holes we showed them to be naked singularities whose existence is questionable by the Weak Cosmic Censorship Hypothesis. Thereby we may assume Schwarzshild black holes to be the only kind created i the LHC, which by the previous arguments are harmless.

Now, one may argue that our theoretical treatment has insufficient credibility due to the fact, that the employed quantum theory is not a well enough description of the physical mechanisms. This is a legitimate counterargument, and we may not rule out the idea that a future theory of quantum gravity may yield new results for our cases. Nevertheless, the sheer scale our our quantitative results are so overwhelmingly large, that even big changes to the theory are unlikely to change the conclusion. It would need to change our result by more than a factor of 10^{50} years to actually cause any concern. Thus, a refined theory should not change the final interpretation of the calculations.

So far we have only been working with the topic on a theoretical and approximate level. Luckily, there is a naturally occurring experiment, that further enforces our already solid conclusion. The LHC creates high energy particle collisions, but cosmic radiation hitting the atmosphere of the earth results in collisions with much higher energy. This has been going on for billions of years. If we assume that these collisions do create dangerous black holes, then it would be unreasonable that the Earth still exists to this day. This yields a contradiction in our primary assumption which is that microscopic black holes are dangerous, or even that they can be created by particle collision.

An interesting extension to this discussion is to consider whether it is possible to observe the presence of a microscopic black hole. If the proton within hydrogen were to collapse into a singularity, one would observe a change in the spectrum of the corresponding electron. This is due to the fact that the proton have some finite size, while the singularity is a point in space time (or a ring for the case of rotating black holes). Thereby, one may use this method to search for microscopic charged black holes, if their existence is possible.

9 Conclusion

Throughout the paper we applied a number of methods to determine the consequences of creating microscopic black holes. Using a numerical solution to the radial equation of motion with imposed conditions we found the classical absorption cross section of a Schwarzschild black hole. From this we considered the case of a Schwarzschild black hole oscillating in the earth yielding a time of 10^{55} years, before it absorbs even a single electron. We then calculated the quantum mechanical absorption cross section which further reinforced this. To improve upon the point, hawking radiation showed the lifespan of such singularities to be of order 10^{-101} seconds.

Thereby we conclude, that the creation of Schwarzschild black holes would be harmless due to the timescales of absorption and their short lifespan. The size of the scales meant even large corrections to the theory would be likely to have no influence on this conclusion. Through a classical treatment of charged or rotating black holes, we found that both cases showed to be naked singularities for such scales. By the weak cosmic censorship hypothesis we may thereby disregard these cases all together.

This conclusion is further reinforced by the existence of cosmic radiation, that at no point yielded deadly singularities. To conclude, black holes created from the LHC will never be any cause for concern.

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