Effect of one omega meson on the deuteron in the nuclear model with explicit mesons

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Preface

The following 15 ECTS bachelor thesis is written so that a third year physics student at Aarhus University should be able to understand it (no additional requirements should be needed).
The idea and research question were provided by my supervisor Dmitri Fedorov, who I planned to have weekly meetings with and who helped me much along the way.
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Abstract

The nuclear model with explicit mesons (MEM) has gained popularity in recent years, reproducing the mass and the width of the N(1440) Roper resonance [1] and the binding energy of the deuteron [2] among other things. The question about how to include repulsion in the model remains unknown. In the following thesis an ω meson has been added to the established deuteron model, trying to explain repulsion.
Through the variational principle and correlated Gaussian's as base functions the binding energy of the deuteron could be reproduced (about -2.20 MeV), but adding the ω meson led to a lowering in energy and therefore not to repulsion. Two different types of creation operators were considered, which introduced 2 new parameters to the model. Unfortunately no reasonable combination of these parameters resulted in an increase in energy. An investigation of the effective potential showed that the addition of the ω meson results in a deeper effective potential and no repulsion could be found. A new approach independent of the variational principle or a different form of creation operator may be needed to explain repulsion in the MEM.

Resumé

En nuklear model med eksplicitte mesoner (MEM) har fået mere popularitet igennem de sidste år. Herved kunne man blandt andet reproducere bindingsenergien af deuteronen [2] og massen og bredden af N(1440) [1]. Det er stadigvæk uklart, hvordan frastødning kan inkluderes i modellen. I det følgende projekt bliver der tilføjet en ω-meson til den etablerede deuteronmodel, som forsøg på at forklare frastødning.
Igennem brugen af variationsprincippet og korrelerede gauss som basisfunktioner kunne den korrekte bindingsenergi af deuteronen reproduceres (omkring -2.20 MeV). Tilføjelsen af ω-mesonen førte til en lavere energi og dermed ikke til frastødning. To forskellige former af skabelsesoperator blev brugt for at danne ω-mesonen, hvorved to nye parametre blev introduceret. Det fremtrådte at ingen realistisk kombination af disse parametre førte til frastødning. Desuden resulterede tilføjelsen af ω-mesonen til et dybere effektiv potentiale, uden tegn på frastødning. En ny tilgang uden variationsprincippet eller en ny form for skabelsesoperator kunne hjælpe med at forklare frastøding i MEM.

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Chapter 1

Introduction

In the nuclear model with explicit mesons (MEM), the nucleons do not interact via a phenomenological potential but rather by the emission and absorption of mesons[2]. The mesons are treated on the same footing as the nucleons and the nucleus is represented by a superposition of states, where the nucleons are surrounded by a different number of mesons [1]. The nucleus is hold together by transitioning between these states[2].
The advantages of this model are the reduced number of parameters, natural inclusion of mesonic physics and natural inclusion of few body forces [2].
In recent years the model had success reproducing the binding energy and the charge radius of the deuteron [2], the mass and the width of the N(1440) Roper resonance [1] and the cross section for neutral pion photoproduction off protons [3] among other things.
However it isn't clear, how repulsion is included in this model. Since the ω meson is responsible for short range repulsion[4], it might be a good starting point. In the following the exchange of an ω meson is investigated and some simple models are looked at, trying to explain repulsion in the MEM.
The model and methods used will be in continuation of [2], where the deuteron could be modeled by a superposition of the nucleons and the nucleons plus a σ meson. In extension of that, the ω meson will

be added and the resulting energy will be compared with the found binding energy of about -2.20 MeV [2]. If adding the ω meson results in an increase in energy, it might indicate repulsion.

Chapter 2

Theory

2.1 System

In the MEM theory, the deuteron can be modeled by a superposition of the bare nucleons and the nucleons with one sigma meson. According to [2] this leads to the correct binding energy. Expanding this form, we are going to add another subsystem with the nucleons and an ω meson. The state is now a superposition of 3 states:

$$\Psi = \begin{pmatrix} \psi_{np}(\vec{r_n}, \vec{r_p}) \\ \psi_{np\sigma}(\vec{r_n}, \vec{r_p}, \vec{r_\sigma}) \\ \psi_{np\omega}(\vec{r_n}, \vec{r_p}, \vec{r_\omega}). \end{pmatrix} \tag{2.1}$$

The advantage of this form is that it can easily be compared with the model including only one sigma meson, where we know what to expect (more to this later).
Looking at the time independent Schrödinger equation leads to these sets of equations:

$$\begin{pmatrix} K_n + K_p & W_\sigma^\dagger & W_\omega^\dagger \\ W_\sigma & K_n + K_p + K_\sigma + m_\sigma & 0 \\ W_\omega & 0 & K_n + K_p + K_\omega + m_\omega \end{pmatrix} \Psi = E\Psi \tag{2.2}$$

, where W is a creation operator, which creates a meson of the given type and $W^\dagger$ annihilates it (we will discuss these later). Note that

we have subtracted the masses of the nucleons from the diagonals of the Hamiltonian. This leads to the energy on the right side being the binding energy directly. We can also write the wavefunction as a linear combination of basis functions[5]:

$$|\Psi\rangle = \sum_{i=1}^{n} c_i |\psi_i(\alpha_i)\rangle. \tag{2.3}$$

, with α_i being a parameter (we will later implement this form through correlated Gaussian's).
Inserting this and multiplying with $\langle\Psi|$ from the left results in:

$$\sum_{i,j=1}^{n} c_i^* c_j \langle\psi_i|H|\psi_j\rangle = E \sum_{i,j=1}^{n} c_i^* c_j \langle\psi_i \,|\, \psi_j\rangle. \tag{2.4}$$

Isolating the energy leads to:

$$E[\Psi] = \frac{c^\dagger \mathcal{H} c}{c^\dagger \mathcal{N} c} \tag{2.5}$$

, where c is an n×1 vector with the coefficients and where

$$\mathcal{H}_{ij} = \langle\psi_i|H|\psi_j\rangle \quad \text{and} \quad \mathcal{N}_{ij} = \langle\psi_i \,|\, \psi_j\rangle. \tag{2.6}$$

2.2 Variational principle

The variational principle is widely used in the MEM theory and we will also use it here. Ritz Theorem states that for an arbitrary function Ψ of the state space the expectation value of H in the state Ψ is such that

$$E \equiv \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi \,|\, \Psi \rangle} \geq E_1 \tag{2.7}$$

, where E_1 is the ground energy (in our case the binding energy) [5]. This approach of cause gives an approximation of the binding energy, but with a good choice of the basis function, we can expect a high chance to come close to the true binding energy. This of course depends on our trail function Ψ, where we are using the form of Ψ as in *eq*. (2.3). A popular choice of basis functions are correlated Gaussian's[6][7][8], which where also used as the base function in [2], when calculating the binding energy of the deuteron. Differentiating equation eq. (2.5) with respect to the coefficients leads to[9]:

$$\frac{\partial E[\Psi]}{\partial c} = \frac{1}{(c^\dagger \mathcal{N} c)^2} \left(c^\dagger \mathcal{N} c \frac{\partial}{\partial c} \left(c^\dagger \mathcal{H} c \right) - c^\dagger \mathcal{H} c \frac{\partial}{\partial c} \left(c^\dagger \mathcal{N} c \right) \right) \tag{2.8}$$

$$= \frac{2}{(c^\dagger \mathcal{N} c)^2} \left(c^\dagger \mathcal{N} c \cdot c^\dagger \mathcal{H} - c^\dagger \mathcal{H} c \cdot c^\dagger \mathcal{N} \right) \tag{2.9}$$

$$= \frac{2}{c^\dagger \mathcal{N} c} \left(c^\dagger \mathcal{H} - E[\Psi] c^\dagger \mathcal{N} \right) \tag{2.10}$$

$$= \frac{2}{c^\dagger \mathcal{N} c} \left(\mathcal{H} c - E[\Psi] \mathcal{N} c \right)^\dagger \tag{2.11}$$

, where we have used, that $\mathcal{H}$ and $\mathcal{N}$ are hermitian. Hence we see that minimizing the energy with respect to c is equivalent to solving the following equation:

$$\mathcal{H} c - E \mathcal{N} c. \tag{2.12}$$

This is the equation we have to solve to find the binding energy. If it rises compared to the system with one σ meson, this might indicate that the ω meson is responsible for repulsion in the MEM.

2.3 Jacobi coordinates

Before implementing the correlated Gaussian's as base functions it is beneficial to make a coordinate transformation. The Jacobi coordinates are a sufficient choice, since they reduce the dimension N of our 3 body problem by one, because the center of mass coordinate can be omitted[1]. They are defined by [10]:

$$\vec{x}_i = \sum_{j=1}^{N} U_{ij}\vec{r}_j \tag{2.13}$$

, where $\mathbf{r} = (\vec{r}_1, ..., \vec{r}_N)$ are single particle coordinates, $\vec{x_N}$ is chosen to be the center of mass coordinate and the rest of the coordinates $\vec{x}_1, ..., \vec{x}_{N-1}$ is a set of independent relative coordinates (see fig. 2.1 for an example). The matrix U is defined as (with $m_{12...i} = m_1 + m_2 + ... + m_i$)[10]:

$$U = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ \frac{m_1}{m_{12}} & \frac{m_2}{m_{12}} & -1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ \frac{m_1}{m_{12...N}} & \frac{m_2}{m_{12...N}} & \dots & \dots & \frac{m_N}{m_{12...N}} \end{pmatrix}. \tag{2.14}$$

,which leads to (the subscript m is used for either meson):

$$\vec{r}_{np} = \vec{r}_p - \vec{r}_n, \qquad \vec{r}_{npm} = \frac{m_n\vec{r}_n + m_p\vec{r}_p}{m_n + m_p} - \vec{r}_m \tag{2.15}$$

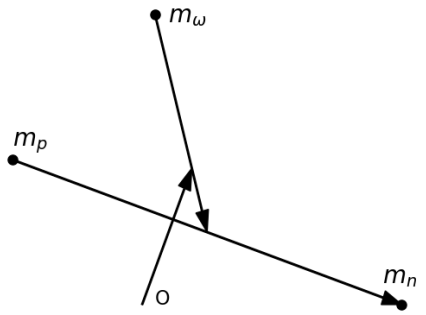


FIGURE 2.1: The Jacobi coordinates for the npω subsystem (not to scale). The vector from the origin O follows the center of mass. The figure is inspired by figure 2.1 in [10].

1: Note that we only have a 3 body problem, because we aren't looking at the subsystem, where there is both an σ and ω meson.

The transformation of the kinetic energy operator with the center-of-mass kinetic energy subtracted is[10]:

$$\sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m_i} - T_{cm} = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \Lambda_{ij} \vec{\pi}_i \vec{\pi}_j \tag{2.16}$$

,with

$$\vec{\pi}_i = -i\hbar \frac{\partial}{\partial \vec{x}_i} \quad \text{and} \quad \Lambda_{ij} = \sum_{k=1}^{N} U_{ik} U_{jk} \frac{1}{m_k}. \tag{2.17}$$

This leads to (see appendix A):

$$K_{np} = -\frac{\hbar^2}{2\mu_{np}} \frac{\partial^2}{\partial \vec{r}_{np}^{\,2}}, \qquad K_{npm} = -\frac{\hbar^2}{2\mu_{npm}} \frac{\partial^2}{\partial \vec{r}_{npm}^{\,2}} \tag{2.18}$$

, where μ is the respective reduced mass:

$$\mu_{np} = \frac{m_n m_p}{m_n + m_p}, \qquad \mu_{npm} = \frac{m_m (m_n + m_p)}{m_n + m_p + m_m}. \tag{2.19}$$

2.4 Correlated Gaussian

We are now ready to expand our states in terms of correlated Gaussian's. We write the wavefunctions as [2]:

$$\psi_{np}(\mathbf{r}^{(d)}) = \sum_{i=1}^{n^{(d)}} c_i^{(d)} e^{-\mathbf{r}^{(d)\mathrm{T}} A_i^{(d)} \mathbf{r}^{(d)}} \equiv \sum_{i=1}^{n^{(d)}} c_i^{(d)} \langle \mathbf{r}^{(d)} \mid A_i^{(d)} \rangle, \tag{2.20}$$

$$\psi_{np\sigma}(\mathbf{r}^{(\sigma)}) = \sum_{j=1}^{n^{(\sigma)}} c_j^{(\sigma)} e^{-\mathbf{r}^{(\sigma)\mathrm{T}} A_j^{(\sigma)} \mathbf{r}^{(\sigma)}} \equiv \sum_{i=1}^{n^{(\sigma)}} c_j^{(\sigma)} \langle \mathbf{r}^{(\sigma)} \mid A_j^{(\sigma)} \rangle, \tag{2.21}$$

$$\psi_{np\omega}(\mathbf{r}^{(\omega)}) = \sum_{k=1}^{n^{(\omega)}} c_k^{(\omega)} e^{-\mathbf{r}^{(\omega)\mathrm{T}} A_k^{(\omega)} \mathbf{r}^{(\omega)}} \equiv \sum_{k=1}^{n^{(\omega)}} c_k^{(\omega)} \langle \mathbf{r}^{(\omega)} \mid A_k^{(\omega)} \rangle. \tag{2.22}$$

, where

$$\mathbf{r}^{(d)} = \left(\vec{r}_{np}\right), \quad \mathbf{r}^{(\sigma)} = \begin{pmatrix} \vec{r}_{np} \\ \vec{r}_{np\sigma} \end{pmatrix} \quad \text{and} \quad \mathbf{r}^{(\omega)} = \begin{pmatrix} \vec{r}_{np} \\ \vec{r}_{np\omega} \end{pmatrix}. \tag{2.23}$$

The upper limit of the sums ($n^{(d)}$, $n^{(\sigma)}$, $n^{(\omega)}$) are the number of Gaussian's in the respective subsystem and ($c_i^{(d)}$,$c_j^{(\sigma)}$,$c_k^{(\omega)}$,$A_i^{(d)}$,$A_j^{(\sigma)}$,$A_k^{(\omega)}$) are variational parameters. The A parameters are chosen stochastically, while the c coefficients are obtained by solving the generalized eigenvalue problem, where the elements can now be written as:

$$\mathcal{H} = \begin{pmatrix} \langle A_i^{(d)} | K_{np} | A_{i'}^{(d)} \rangle & \langle A_i^{(d)} | W_\sigma^\dagger | A_j^{(\sigma)} \rangle & \langle A_i^{(d)} | W_\omega^\dagger | A_k^{(\omega)} \rangle \\ \langle A_j^{(\sigma)} | W_\sigma | A_i^{(d)} \rangle & \langle A_j^{(\sigma)} | K_{np} + K_{np\sigma} + m_\sigma | A_{j'}^{(\sigma)} \rangle & 0 \\ \langle A_k^{(\omega)} | W_\omega | A_i^{(d)} \rangle & 0 & \langle A_k^{(\omega)} | K_{np} + K_{np\omega} + m_\omega | A_{k'}^{(\omega)} \rangle \end{pmatrix} \tag{2.24}$$

and

$$\mathcal{N} = \begin{pmatrix} \langle A_i^{(d)} \mid A_{i'}^{(d)} \rangle & 0 & 0 \\ 0 & \langle A_j^{(\sigma)} \mid A_{j'}^{(\sigma)} \rangle & 0 \\ 0 & 0 & \langle A_k^{(\omega)} \mid A_{k'}^{(\omega)} \rangle \end{pmatrix}. \tag{2.25}$$

Consequently we have to solve a n×n eigenvalue problem (where n is the number of combined Gaussian), where the matrices can be written as 3×3 block matrices.

2.4.1 Notation

A new notation has been introduced above, namely that vectors written in bold contain other vectors, while 3-dimensional vectors are written with a vector arrow. For example:

$$\mathbf{r}^{(\omega)} = \begin{pmatrix} \vec{r}_{np} \\ \vec{r}_{np\omega} \end{pmatrix}. \tag{2.26}$$

Note that, while $\mathbf{r}^{(d)}$ actually contains only one 3-d vector, the bold notation is kept for consistency. Apart from that we'll donate vectors that pick out single 3-d vectors as w_i:

$$w_i^{\mathrm{T}}\mathbf{r} = \vec{r}_i. \tag{2.27}$$

These new bold vectors behave as follows [6]:

$$\mathbf{a}^{\mathrm{T}}\mathbf{b} = \sum_{i=1}^{N} \vec{a}_i \cdot \vec{b}_i, \tag{2.28}$$

$$\mathbf{a}^{\mathrm{T}}A\mathbf{b} = \sum_{i=1}^{N}\sum_{j=1}^{N} \vec{a}_i \cdot \vec{b}_i A_{ij}. \tag{2.29}$$

The gradient in this notation becomes[9]:

$$\frac{\partial}{\partial \mathbf{r}} = \left(\frac{\partial}{\partial \vec{r}_1}, \frac{\partial}{\partial \vec{r}_2}, ..., \frac{\partial}{\partial \vec{r}_N} \right). \tag{2.30}$$

Going back to our wavefunctions, we see that we can write the Gaussian part as[2]:

$$\langle \mathbf{r} \,|\, A \rangle = \exp\left(-\mathbf{r}^{\mathrm{T}}A\mathbf{r}\right) = \exp\left(-\sum_{i,j=1}^{N} \left(\frac{\vec{r}_i - \vec{r}_j}{b_{ij}} \right)^2 \right) \tag{2.31}$$

, with the matrix A being:

$$A = \sum_{i<j=1}^{N} \frac{w_{ij} w_{ij}^{\mathrm{T}}}{b_{ij}^2} \tag{2.32}$$

and w_{ij} being defined by:

$$w_{ij}^{\mathrm{T}}\mathbf{r} = \vec{r}_i - \vec{r}_j. \tag{2.33}$$

Under the Jacobi transformation, these transform as [2]:

$$w_{ij} \rightarrow (U^{-1})^{\mathrm{T}} w_{ij} \tag{2.34}$$

,which follows from:

$$w_{ij}^{\mathrm{T}}\mathbf{r} = w_{ij}^{\mathrm{T}}(U^{-1})\mathbf{x} = ((U^{-1})^{\mathrm{T}} w_{ij})^{\mathrm{T}}\mathbf{x}. \tag{2.35}$$

The parameters b_{ij} are range parameters and are chosen stochastically from:

$$b_{ij} = -\ln(u)b \tag{2.36}$$

, where u is a pseudo random number and b is a range parameter.
We now have a way to calculate the A matrices, which are needed for calculating the matrix elements later on. The calculation can be found in appendix B. The results are:

$$A^{(d)} = \frac{1}{b_1^2} \tag{2.37}$$

and

$$A^{(m)} = \frac{1}{b_1^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{b_2^2}\begin{pmatrix} \frac{m_p^2}{(m_n+m_p)^2} & \frac{m_p}{m_n+m_p} \\ \frac{m_p}{m_n+m_p} & 0 \end{pmatrix} + \frac{1}{b_3^2}\begin{pmatrix} \frac{m_n^2}{(m_n+m_p)^2} & \frac{m_n}{m_n+m_p} \\ \frac{m_n}{m_n+m_p} & 0 \end{pmatrix} \tag{2.38}$$

, where m is used for either meson. Note that $A^{(d)}$ actually just becomes a scalar, but we'll again keep the notation for consistency.

2.5 W operator

Before calculating the matrix elements, we have to know the creation operator (W) and the corresponding annihilation operator ($W^\dagger$). Fortunately from [2] we already know a creation and annihilation operator for the σ meson that works, namely:

$$W_\sigma = A \cdot S_\sigma \cdot \exp\left(-\frac{\vec{r}_{np}^2 + \vec{r}_{np\sigma}^2}{b_\sigma^2}\right) \tag{2.39}$$

,where S_σ and b_σ are parameters of the model and A is a normalization factor, which can be pulled into S_σ. In [2] $S_\sigma = 20.35$ MeV and $b_\sigma = 3$ fm could reproduce the binding energy and charge radius of the deuteron and we will keep them here. The form of this operator is chosen very well, since it will vanish if either of the 3 particles is far away.
We will now start to find the simplest and consistent form of the W_ω operator. We know that the interaction has to obey some conservation laws. In our case the most important ones are the conservation of angular momentum (which corresponds to spin, since we're just looking at s waves), the conservation of isospin and the conservation of parity. Looking at the matrix element of the form:

$$\langle A_k^{(\omega)} | W | A_i^{(d)} \rangle \tag{2.40}$$

we realize, that both the bra and ket are scalars. This leads to the conclusion, that the W operator also should be a scalar and therefore should have positive parity. Furthermore we of course know that the ω meson should be included. The ω meson has negative parity, is an isoscalar and a vector meson (spin=1). Since it is already an isoscalar we don't have to do anything there, but we have to pair it with a vector with negative parity. The simplest operator of this form is:

$$W_\omega = (\vec{\omega} \cdot \vec{r}_{np\omega}) F(\mathbf{r}^{(\omega)}) \tag{2.41}$$

,where the function $F(\mathbf{r}^{(\omega)})$ is a short-range form-factor of the same form as before:

$$F(\mathbf{r}^{(\omega)}) = A \cdot S_\omega \cdot \exp\left(-\frac{\vec{r}_{np}^2 + \vec{r}_{np\omega}^2}{b_\omega^2}\right). \tag{2.42}$$

In this case A is again a normalization constant and S_ω and b_ω are new parameters of the model. The normalization constant is again ignored, since it can be pulled into the S_ω constant.
Looking at the lower equation of the Schödinger equation leads to the following equation:

$$(\vec{\omega} \cdot \vec{r}_{np\omega})F(\mathbf{r}^{(\omega)})\psi_{np} + (K_{np} + K_{np\omega} + m_\omega)\psi_{np\omega} = E\psi_{np\omega} \tag{2.43}$$

, suggesting that the wavefunction of the subsystem with one ω meson has the form (also known as prefactor Gaussian):

$$\psi_{np\omega} = (\mathbf{a}^{\mathrm{T}} \cdot \mathbf{r}^{(\omega)}) \sum_{k=1}^{n^{(\omega)}} c_k^{(\omega)} e^{\mathbf{r}^{(\omega)\mathrm{T}} A_k^{(\omega)} \mathbf{r}^{(\omega)}}. \tag{2.44}$$

Note that both the spin and isospin part of the wavefunction haven't been written down, since they don't matter (because our operators aren't acting on either spin or isospin).
Just like $\vec{\omega}$, we require that:

$$\mathbf{a}^{\mathrm{T}}\mathbf{a} = (c_1\vec{a}_1, c_2\vec{a_2}) \begin{pmatrix} c_1\vec{a}_1 \\ c_2\vec{a}_2 \end{pmatrix} = c_1^2\vec{a}_1^{\,2} + c_2^2\vec{a}_2^{\,2} = 1. \tag{2.45}$$

That $\vec{a}_i^{\,2} = 1$ can easily be achieved by introducing spherical coordinates:

$$\vec{a}_i = \begin{pmatrix} \sin(\theta_i)\cos(\phi_i) \\ \sin(\theta_i)\sin(\phi_i) \\ \cos(\theta_i) \end{pmatrix} \tag{2.46}$$

, which together with:

$$c_1 = \cos(\chi) \quad \text{and} \quad c_2 = \sin(\chi) \tag{2.47}$$

fulfills the normalization condition. The new variables θ_i, ϕ_i and χ are also treated as variational parameters, which are created using pseudo random numbers.
Apart from that we'll also look at a different form of the creation operator, namely:

$$W_\omega = (\vec{\omega} \cdot \vec{r}_{np\omega})^2 F(\mathbf{r}^{(\omega)}) \tag{2.48}$$

, which leads to:

$$\psi_{np\omega} = (\mathbf{a}^{\mathrm{T}} \cdot \mathbf{r}^{(\omega)})(\mathbf{b}^{\mathrm{T}} \cdot \mathbf{r}^{(\omega)}) \sum_{k=1}^{n^{(\omega)}} c_k^{(\omega)} e^{\mathbf{r}^{(\omega)\mathrm{T}} A_k^{(\omega)} \mathbf{r}^{(\omega)}}. \tag{2.49}$$

The form of these wavefunctions will come very handy when looking at the matrix elements, which we are going to do next.

2.6 Matrix elements

Fortunately the matrix elements of the matrices are all analytic, which is one of the big advantages of the correlated Gaussian method.
The calculation of these matrix elements can be quite long. Since they have been calculated before [11][6], we will simple state the relevant results, starting with the diagonal terms without kinetic energy:

$$\langle B \,|\, A\rangle \equiv M_0 = \left(\frac{\pi^n}{\det(A+B)}\right)^{3/2}, \tag{2.50}$$

$$\langle B|(\mathbf{b}^{\mathrm{T}}\mathbf{r})(\mathbf{a}^{\mathrm{T}}\mathbf{r})|A\rangle \equiv M_1 = \frac{1}{2}(\mathbf{b}^T R\mathbf{a})M_0, \tag{2.51}$$

$$\langle B|(\mathbf{d}^{\mathrm{T}}\mathbf{r})(\mathbf{c}^{\mathrm{T}}\mathbf{r})(\mathbf{b}^{\mathrm{T}}\mathbf{r})(\mathbf{a}^{\mathrm{T}}\mathbf{r})|A\rangle \equiv M_2 \tag{2.52}$$

$$= \frac{1}{4}\left[(\mathbf{b}^{\mathrm{T}}R\mathbf{a})(\mathbf{d}^{\mathrm{T}}R\mathbf{c}) + (\mathbf{b}^{\mathrm{T}}R\mathbf{c})(\mathbf{d}^{\mathrm{T}}R\mathbf{a}) + (\mathbf{b}^{\mathrm{T}}R\mathbf{d})(\mathbf{c}^{\mathrm{T}}R\mathbf{a})\right]M_0$$

, where n is the dimension of either A or B and $R = (A+B)^{-1}$.
Now let's have a look at the terms with kinetic energy, where we write the kinetic energy in the general form[12]:

$$\hat{K} = -\sum_{i,j=1}^{N} \frac{\partial}{\partial \vec{r}_i} K_{ij} \frac{\partial}{\partial \vec{r}_j} \equiv -\frac{\partial}{\partial \mathbf{r}} K \frac{\partial}{\partial \mathbf{r}^{\mathrm{T}}}. \tag{2.53}$$

With eq. (2.30) it becomes trivial to see that the matrices of the given subsystems become:

$$K^{(d)} = \left(\tfrac{\hbar^2}{2\mu_{np}}\right), \quad K^{(\sigma)} = \begin{pmatrix} \frac{\hbar^2}{2\mu_{np}} & 0 \\ 0 & \frac{\hbar^2}{2\mu_{np\sigma}} \end{pmatrix}, \quad K^{(\omega)} = \begin{pmatrix} \frac{\hbar^2}{2\mu_{np}} & 0 \\ 0 & \frac{\hbar^2}{2\mu_{np\omega}} \end{pmatrix}. \tag{2.54}$$

The matrix elements with the kinetic energy are:

$$\langle B|\hat{K}|A\rangle = 6\mathrm{Tr}(BKAR)M_0 \equiv LM_0, \tag{2.55}$$

$$\langle B|(\mathbf{b}^{\mathrm{T}}\mathbf{r})\hat{K}(\mathbf{a}^{\mathrm{T}}\mathbf{r})|A\rangle = LM_1 + \mathbf{b}^{\mathrm{T}}K'\mathbf{a}M_0 \tag{2.56}$$

$$\langle B|(\mathbf{d}^{\mathrm{T}}\mathbf{r})(\mathbf{c}^{\mathrm{T}}\mathbf{r})\hat{K}(\mathbf{b}^{\mathrm{T}}\mathbf{r})(\mathbf{a}^{\mathrm{T}}\mathbf{r})|A\rangle = LM_2 \tag{2.57}$$

$$+\frac{M_0}{2}\left[(\mathbf{a}^{\mathrm{T}}R\mathbf{c})(\mathbf{d}^{\mathrm{T}}K'\mathbf{b}) + (\mathbf{a}^{\mathrm{T}}R\mathbf{d})(\mathbf{c}^{\mathrm{T}}K'\mathbf{b})\right.$$

$$+(\mathbf{b}^{\mathrm{T}}R\mathbf{c})(\mathbf{d}^{\mathrm{T}}K'\mathbf{a}) + (\mathbf{b}^{\mathrm{T}}R\mathbf{d})(\mathbf{c}^{\mathrm{T}}K'\mathbf{a})$$

$$\left.-(\mathbf{a}^{\mathrm{T}}R\mathbf{b})(\mathbf{d}^{\mathrm{T}}K_1\mathbf{c}) - (\mathbf{c}^{\mathrm{T}}R\mathbf{d})(\mathbf{b}^{\mathrm{T}}K_2\mathbf{a})\right]$$

,with

$$K' = (1 - RB)K(1 - AR) + RAKBR), \tag{2.58}$$

$$K_1 = (1 - RB)KAR + RAK(1 - BR), \tag{2.59}$$

$$K_2 = (1 - RA)KBR + RBK(1 - AR). \tag{2.60}$$

Now the only matrix elements left are the off diagonals. The $\langle A_j^{(\sigma)}|W_\sigma|A_i^{(d)}\rangle$ elements were already calculated in [2]:

$$\langle A_j^{(\sigma)}|W_\sigma|A_i^{(d)}\rangle = S_\sigma\langle A_j^{(\sigma)}\,|\,\tilde{A}_i\rangle \tag{2.61}$$

, with $\tilde{A}_i$ given as:

$$\tilde{A}_i = \begin{pmatrix} A_i^{(d)} + \frac{1}{b_\sigma^2} & 0 \\ 0 & \frac{1}{b_\sigma^2}. \end{pmatrix} \tag{2.62}$$

We will use the same approach for the other matrix elements:

$$\langle A_k^{(\omega)}|W_\omega|A_i^{(d)}\rangle = S_\omega\langle A_k^{(\omega)}|(\mathbf{a}^{\mathrm{T}}\mathbf{r}^{(\omega)})(\omega^{\mathrm{T}}\mathbf{r}^{(\omega)})|\bar{A}_i\rangle \tag{2.63}$$

, with:

$$\omega = \begin{pmatrix} \vec{0} \\ \vec{\omega} \end{pmatrix} \quad \text{and} \quad \bar{A}_i = \begin{pmatrix} A_i^{(d)} + \frac{1}{b_\omega^2} & 0 \\ 0 & \frac{1}{b_\omega^2}. \end{pmatrix}. \tag{2.64}$$

Now the matrix elements can be calculated with eq. (2.51) above.
The form of $\tilde{A}_i$ and $\bar{A}_i$ originates from the short-range form-factor part of the creation operator. In the following the subscript m is again used for either meson. Looking at the matrix elements we have:

$$\langle A_j^{(m)}|W_m|A_i^{(d)}\rangle \propto \langle A_j^{(m)}|e^{-\frac{\vec{r}_{np}^2+\vec{r}_{npm}^2}{b_m^2}}\,e^{-\frac{\vec{r}_{np}^2}{b_i^2}}. \tag{2.65}$$

Introducing the matrix notation and $\mathbf{r}^{(m)} = (\vec{r}_{np}, \vec{r}_{npm})^{\mathrm{T}}$ it becomes clear, that we can merge these exponential together into the form $\exp(-\mathbf{r}^{(m)}A_i\mathbf{r}^{(m)})$, with the matrix being:

$$A_i = \begin{pmatrix} A_i^{(d)} + \frac{1}{b_m^2} & 0 \\ 0 & \frac{1}{b_m^2} \end{pmatrix}. \tag{2.66}$$

The same from can be used for the creation operator of the squared form, where the elements are:

$$\langle A_k^{(\omega)}|W_\omega|A_i^{(d)}\rangle = S_\omega\langle A_k^{(\omega)}|(\mathbf{a}^{\mathrm{T}}\mathbf{r}^{(\omega)})(\mathbf{b}^{\mathrm{T}}\mathbf{r}^{(\omega)})(\omega^{\mathrm{T}}\mathbf{r}^{(\omega)})(\omega^{\mathrm{T}}\mathbf{r}^{(\omega)})|\bar{A}_i\rangle \quad (2.67)$$

and eq. (2.52) can be used to calculate the matrix elements.
Since these matrix elements are completely real it follows that:

$$\langle A_k^{(\omega)}|W_\omega|A_i^{(d)}\rangle = \langle A_i^{(d)}|W_\omega^\dagger|A_k^{(\omega)}\rangle \quad (2.68)$$

and the same concludes for the σ meson elements.
Now we are able to see the advantage of constructing the system in the way it has been done, since we can remove the effect of the ω meson by setting S_ω=0 (we then should be able to reproduce the foundings of [2]).

2.7 Numerical methods

Before continuing with the results, the numerical methods used will be described briefly.
First the matrix elements are calculated from the equations above and afterwards assembled into $\mathcal{H}$ and $\mathcal{N}$. Since these are real and symmetric by construction, `scipy.linalg.eigh` can be used to solve the eigenvalue problem (the variational parameters are constructed with eq. (2.36), where the range parameter b is set to: 3 fm for the np and npσ subsystem, 1.5 fm for the npω subsystem and 1 for all angles).
The solution is now minimized with the `scipy.optimize.minimize` function, where the conjugate gradient method is used. Trying to ensure that we are finding the global minimum, the minimization process is divided into 3 iteration with different tolerances. As starting point a tolerance of 0.06 is used, trying to find the vicinity of the global minimum. Subsequently a tolerance of 0.02 and 0.01 is used to get a more precise result.
Another important point regarding the minimization process is to keep in mind that not all parameters are optimized at once. In each iteration first the $A^{(d)}$ parameters are optimized, then the $A^{(\sigma)}$ (where each $A_j^{(\sigma)}$ gets optimized one after the other), followed by the angles and finally $A^{(\omega)}$ and the angles again (the angles are optimized 2 times in one iteration as achieving convergence was more difficult). You hence might get a small error of optimizing them one after each other, but the effect should be minimal since we are iterating 3 times. This means that they can also be refined again after the rest has been optimized too. Apart from that we're not that interested in the exact value, but rather if the energy rises or falls when adding the ω meson.

Chapter 3

Results

3.1 Simplest creation operator

The masses of the nucleons and mesons where taken from [13]: $m_n = 939.565$ MeV, $m_p = 938.272$ MeV, $m_\sigma = 500$ MeV and $m_\omega = 782.66$ MeV. When creating the ω meson, we have 3 possibilities, because the ω meson can be created with 3 different spin directions. Since the ω meson gets dotted with $\vec{a}_i$ (which is free to change, since it only depends on variational parameters) it shouldn't matter which spin direction we choose. This is also what we get when changing the spin direction and in the following plots $\omega_z = 1$ is chosen[1], so that the number of plots can be reduced (since $\vec{a_{iz}}$ only depends on 2 angles instead of 3 the matrix elements should converge faster).
Apart from that the model has 4 free parameters: S_σ, b_σ, S_ω and b_ω. Both S_σ and b_σ were taken from [2], since they produced the correct binding energy and charge radius of the deuteron. The values of S_ω and b_ω had to be guessed, since there is no final values to compare to. One may assume, that b_ω (as being a range parameter) is smaller than the corresponding σ parameter, since the mass of the ω meson is much greater compared to the σ meson. As seen in fig. 3.1 it doesn't seem

1: For the interested reader some plots with different spin directions of the ω meson are included in appendix C.

to matter, which parameter values we choose, since all reasonable guesses lead to a decrease in energy (the maximum of ΔE_0 is about -0.0055 MeV). In the following b_ω=1.4 fm and $S_\omega = 20$ MeV are used, since it leads to a clearly recognizable lowering of the energy, but it doesn't get lowered so much, that the figures get too big.

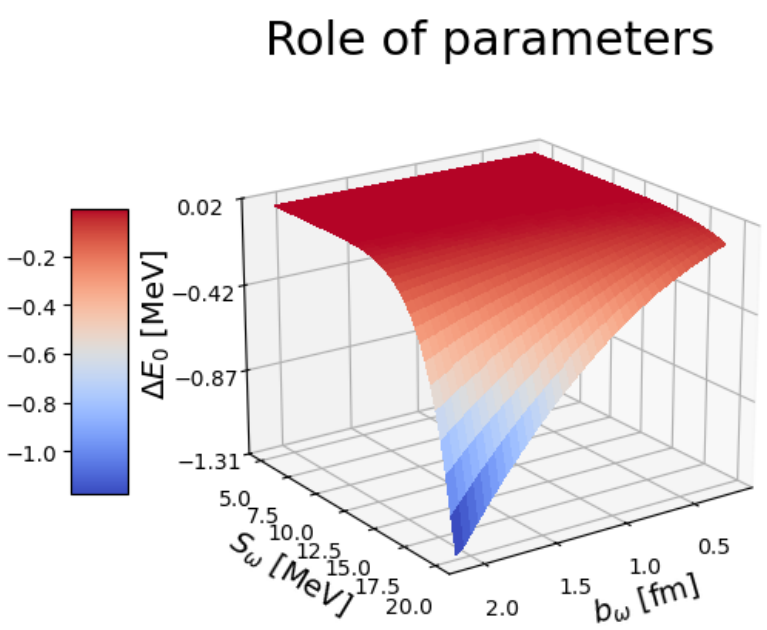
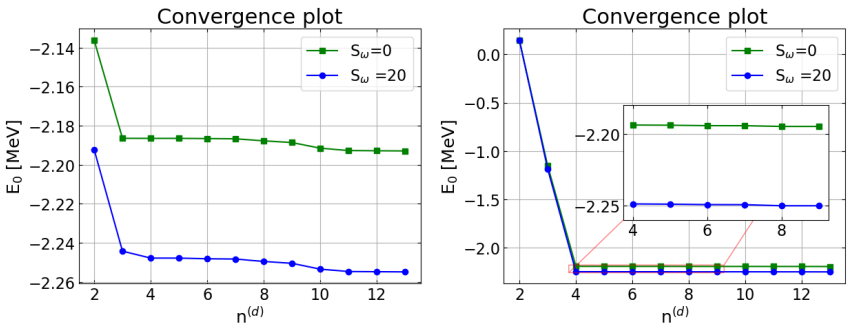


FIGURE 3.1: The difference in binding energy ($\Delta E_0 = E_{0\omega} - E_0$, where E_0 is the binding energy without the ω meson) is plotted with respect to the 2 free parameters: b_ω and S_ω. The number of Gaussian's in the respective subsystems are: $n^{(d)}$=4, $n^{(\sigma)}$=6 and $n^{(\omega)}$=1. The simplest form of creation operator has been used.

Of course when looking at fig. 3.1 we need to have a sufficient amount of Gaussian's in the respective subsystems in order to achieve energy converges. To ensure this, the number of Gaussian's is varied within a subsystem, while holding the rest fixed. Since the ω meson is very heavy, the energy should converge rather fast. Consequently $n^{(\omega)}$ is always set to 1. The results can be seen in fig. 3.2(a) and fig. 3.2(b). In fig. 3.2(a) $n^{(\sigma)}$=6 has been used and it is recognizable, that the energy converges after 3 Gaussian's in the np subsystem. When looking

at fig. 3.2(b), where $n^{(d)} = 4$ has been used, the energy converges after 4 Gaussian's in the npσ subsystem. In fig. 3.1 $n^{(d)}$=4, $n^{(\sigma)}$=6 and $n^{(\omega)}$=1 has been used, since the energy converges for these numbers of Gaussian's.
Furthermore when looking at fig. 3.2(a) and fig. 3.2(b) we notice that the binding energy of about -2.2 MeV could be reproduced for the system without the ω meson. Adding the meson led to a lowering in energy.



(a) Convergence of $n^{(d)}$ (b) Convergence of $n^{(\sigma)}$

FIGURE 3.2: The convergence of the energy for different number of Gaussian's in the respective subsystem with (S_ω=20 MeV) and without (S_ω=0 MeV) the ω meson (b_ω is set to 1.4 fm). In the left figure (a) $n^{(d)}$ was varied, with $n^{(\sigma)} = 6$ and $n^{(\omega)} = 1$ fixed. In the right figure (b), $n^{(\sigma)}$ was varied, with $n^{(d)} = 4$ and $n^{(\omega)} = 1$ fixed. The simplest form of the creation operator has been used.

It looks like we don't have any repulsion, but maybe the wavefunction will give us more insight (maybe we can find a hint of repulsion through the effective potential producing the wavefunction). As seen in fig. 3.3: u(r) with $S_\omega = 60$ MeV is shifted a bit to the left with respect to $S_\omega = 0$ MeV. (The scale parameter S_ω is set to 60 MeV, so that the effect can be seen). Comparing the effective potential of the form [2]:

$$V_{\text{eff}} = E_0 + \frac{\hbar^2}{2\mu_{np}} \frac{u''}{u} \tag{3.1}$$

results in the lower part of fig. 3.3. Adding the ω meson led to a deeper effective potential and no sign of repulsion could be found.

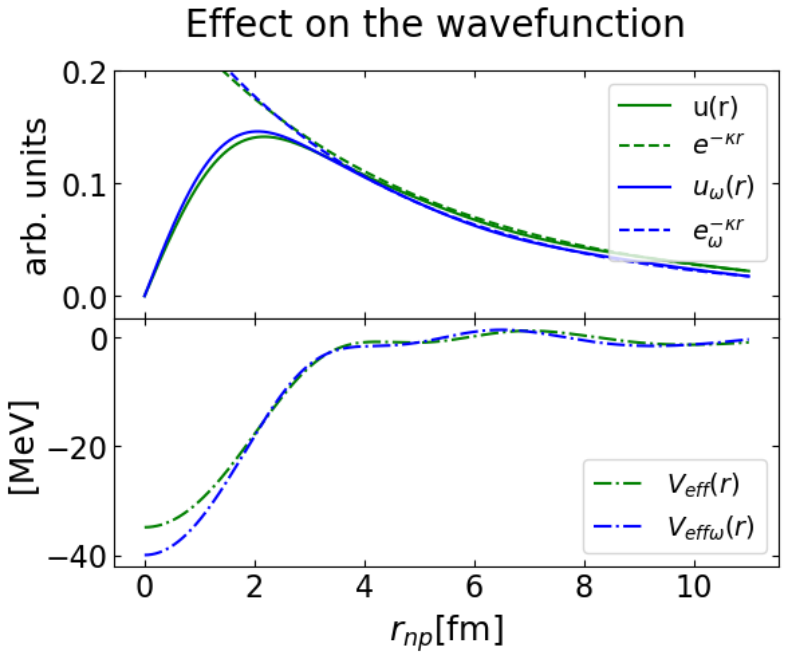


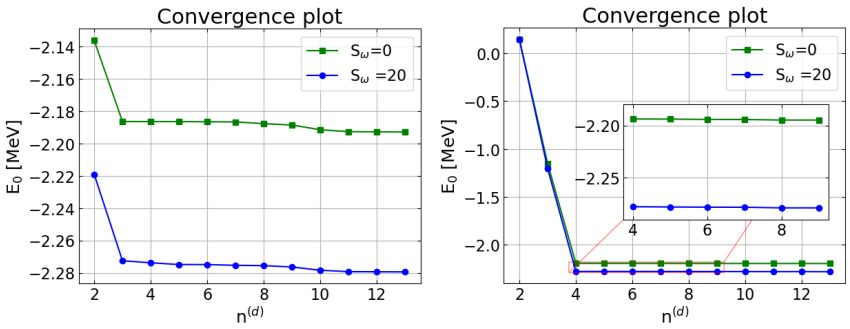
FIGURE 3.3: In the top part u(r)=r·ψ_{np} is plotted with respect to r_{np} together with its asymptotic form $e^{-\kappa r}$, with $\kappa = \sqrt{2\mu_{np}|E_0|/\hbar^2}$[2]. In the lower part of the figure the different effective potentials are compared. The coefficients and parameters needed for the wavefunction were taken from a minimization process with $n^{(d)} = 4$, $n^{(\sigma)} = 6$ and $n^{(\omega)} = 1$. b_ω was set to 1.4 fm and $S_\omega = 0$ MeV is compared to $S_\omega = 60$ MeV. The simplest form of creation operator has been used.

3.2 Squared form

Moving on to the second form of the creation operator, namely:

$$W_\omega = (\vec{\omega} \cdot \vec{r}_{np\omega})^2 F(\mathbf{r}^{(\omega)}) \tag{3.2}$$

, we again have to choose the spin direction of the ω meson. Again it doesn't make a difference and $\omega_z = 1$ is chosen in the following plots[2]. Looking at the convergence and choosing the same parameters as before leads to fig. 3.4(a) and fig. 3.4(b). We reach convergence for the same number of Gaussians's as before in both subplots (which of course isn't a big surprise, since the form of the omega creation operator doesn't influence the convergence when $S_\omega = 0$ MeV and the ω meson is so heavy that $n^{(\omega)} = 1$ is sufficient). Comparison of these convergence plots with the ones before, lead to the conclusion that the energy gets lowered even further with the squared form of the operator.



(a) Convergence of $n^{(d)}$ (b) Convergence of $n^{(\sigma)}$

FIGURE 3.4: The convergence of the energy for different number of Gaussian in the respective subsystem with (S_ω=20 MeV) and without (S_ω=0 MeV) the ω meson (b_ω is set to 1.4 fm). In the left figure (a) $n^{(d)}$ was varied, with $n^{(\sigma)} = 6$ and $n^{(\omega)} = 1$ fixed. In the right figure (b), $n^{(\sigma)}$ was varied, with $n^{(d)} = 4$ and $n^{(\omega)} = 1$ fixed. The squared form of the creation operator has been used.

One might still wonder, if there exists a combination of S_ω and b_ω that leads to repulsion. The conclusion from fig. 3.5 becomes that no

2: For the interested reader some plots with different spin directions of the ω meson are included in appendix C.

combination could be found and that the reduction of energy is again larger for the squared form of the creation operator (the maximum value of ΔE_0 is about -0.0062 MeV).

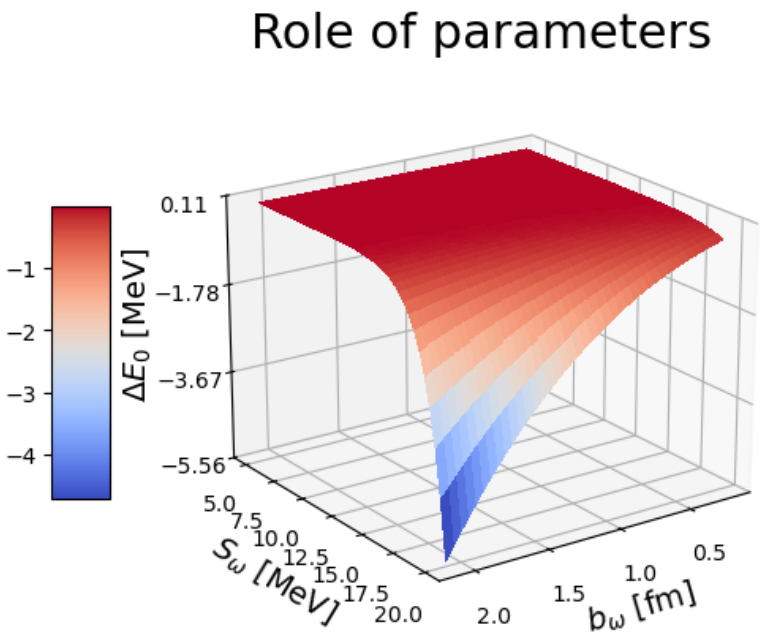


FIGURE 3.5: The difference in binding energy ($\Delta E_0 = E_{0\omega} - E_0$, where E_0 is the binding energy without the ω meson) is plotted with respect to the 2 free parameters: b_ω and S_ω. The number of Gaussian's in the respective subsystems are: $n^{(d)}$=4, $n^{(\sigma)}$=6 and $n^{(\omega)}$=1. The squared form of the creation operator has been used.

Continuing with the the wavefunction (fig. 3.6), we again see a slight shift of u(r) to the left when adding the ω meson (S_ω is again set to 60 MeV, so that the effect can be seen). Looking at the lower part of the figure one concludes that the effective potential becomes deeper (even lower than for the non squared form of the creation operator) and once more no repulsion can be detected.

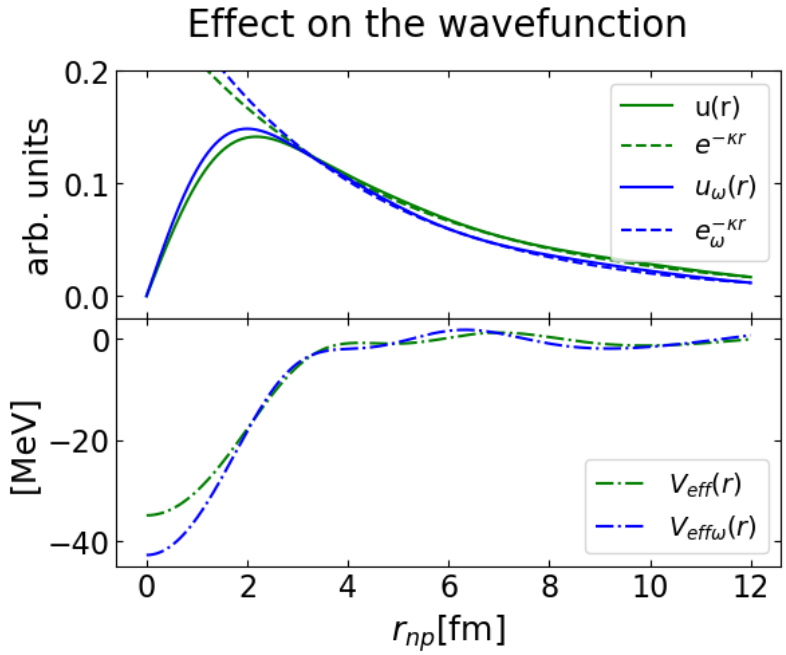
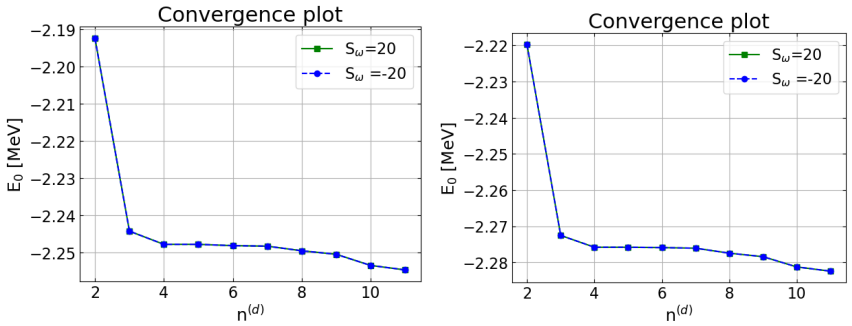


FIGURE 3.6: In the top part u(r)=r·ψ_{np} is plotted with respect to r_{np} together with its asymptotic form $e^{-\kappa r}$. In the lower part of the figure the different effective potentials are compared. The coefficients and parameters needed for the wavefunction were taken from a minimization process with $n^{(d)} = 4$, $n^{(\sigma)} = 6$ and $n^{(\omega)} = 1$. b_ω was set to 1.4 and $S_\omega = 0$ MeV is compared to $S_\omega = 60$ MeV. $W_\omega = (\vec{\omega} \cdot \vec{r}_{np\omega})^2 F(\mathbf{r}^{(\omega)})$ has been used.

3.3 Negative S_ω

Another approach is the application of a negative value for S_ω, which results in the following:



(a) Non squared form (b) Squared form

FIGURE 3.7: The convergence for different number of Gaussian in the np subsystem with (S_ω=20 MeV) and without (S_ω=-20 MeV) the ω meson (b_ω is set to 1.4 fm). $n^{(\sigma)} = 6$ and $n^{(\omega)} = 1$ are fixed, while $n^{(d)}$ is varied. In the left figure (a) the simplest form of the creation operator is used and in figure (b) the squared form is used.

As seen in fig. 3.7(a) and fig. 3.7(b) the sign of S_ω doesn't change anything, since the lines coincide. This could have been anticipated to begin with, by looking at the Hamiltonian of the system:

$$\begin{pmatrix} K_{np} & W_\sigma^\dagger & W_\omega^\dagger \\ W_\sigma & K_{np} + K_{np\sigma} + m_\sigma & 0 \\ W_\omega & 0 & K_{np} + K_{np\omega} + m_\omega \end{pmatrix}. \tag{3.3}$$

When calculating the eigenvalues and therefore the determinant, we recognize that the W_ω term only appears squared. Hence the sign on S_ω shouldn't matter.

Chapter 4

Discussion

In the following we want to discuss the possible reasons why we're only seeing a lowering in energy and what could be the next steps, trying to explain repulsion in the MEM.
First the overall strategy used to answer the question of repulsion should be critically examined. In the MEM theory the variational principle and the use of correlated Gaussian is very popular (see for example [2][11]), but this might cause some problems when trying to find repulsion and hence an increase in energy. Assuming that the ω meson would cause repulsion, one might expect that the minimization process would choose the variational parameters in such a way that the contribution of the ω meson vanishes. Then we wouldn't be able to see an increase in energy, but merely an unchanged energy.
A different reason for not discovering repulsion could of course be the form of the creation operator, where different forms could be tried as the next step. One may for example try to introduce complex numbers, since this changes the sign on the $W^\dagger$ operator. A different opportunity would be to introduce spin dependence by including the Pauli vector in the creation operator. A suitable form would be:

$$W_\omega = (\vec{\omega} \cdot \vec{\sigma})(\vec{\sigma} \cdot \vec{r}_{np\omega})F(\mathbf{r}^{(\omega)}) \tag{4.1}$$

, where $\vec{\sigma}$ donates the Pauli vector.
The creation operator of the the form:

$$W_\omega = \left(\vec{\omega} \cdot \frac{\partial}{\partial \vec{r}_{np\omega}} \right) F(\mathbf{r}^{(\omega)}) \tag{4.2}$$

might be interesting to look at as well.
Another approach could be to include the state with both the σ and ω meson like this:

$$\Psi = \begin{pmatrix} \psi_{np}(\vec{r_n}, \vec{r_p}) \\ \psi_{np\sigma}(\vec{r_n}, \vec{r_p}, \vec{r_\sigma}) \\ \psi_{np\sigma\omega}(\vec{r_n}, \vec{r_p}, \vec{r_\sigma}, \vec{r_\omega}). \end{pmatrix} \tag{4.3}$$

with the Hamiltonian:

$$H = \begin{pmatrix} K_{np} & W_\sigma^\dagger & 0 \\ W_\sigma & K_{np} + K_{np\sigma} + m_\sigma & W_\omega^\dagger \\ 0 & W_\omega & K_{np} + K_{np\sigma} + K_{np\omega} + m_\sigma + m_\omega \end{pmatrix}. \tag{4.4}$$

Another possibility could be the implementation of both $\psi_{np\omega}$ and $\psi_{np\sigma\omega}$ or to simply look at a different vector meson (or exchanging the σ meson with the pion).
Last but not least one could of course explain the absence of repulsion by simply concluding that something might be missing in the MEM, but this should be done last.

Chapter 5

Conclusion

In this thesis we tried to explain repulsion by adding an ω meson to the established deuteron model in the MEM which contains the bare nucleons and a σ meson.
The variational principle together with correlated Gaussian's with prefactors were used to calculate the binding energy. The convergence of the energy could be achieved with 4 Gaussian's in the np subsystem, 6 Gaussian's in the npσ subsystem and 1 Gaussian in the npω subsystem.
When adding the ω meson, two different types of creation operator were considered, which introduced two new parameters b_ω and S_ω. The binding energy of the deuteron (about -2.20 MeV) could be reproduced, while ignoring the ω meson. Adding it resulted in a lowering of the energy. This was true for both types of operators (the squared form resulted in a lower energy) and apparently for all reasonable choices of the parameters b_ω and S_ω (including negative S_ω). Comparing the different effective potentials resulting in an even deeper V_{eff}, when adding the ω meson. Furthermore it did not support repulsion.
New types of creation operators, a new structure of the system or even different methods than the variational method may be the next approaches in trying to explain repulsion in the MEM. If these attempts don't lead to the desired results it might be that something is missing in the model used.

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Appendix A

Kinetic energy transformation

In the following we are going to look at the kinetic energy transformation of the subsystem with one meson m (the subsystem with just the nucleons is identic to just the first term of the subsystem with the meson). Starting with calculating the Λ elements (eq. (2.17)) leads to:

$$\begin{aligned}\Lambda_{11} &= \frac{U_{11}U_{11}}{m_n} + \frac{U_{12}U_{12}}{m_p} \qquad (A.1)\\ &= \frac{1}{m_n} + \frac{1}{m_p} = \frac{m_n + m_p}{m_n m_p} = \frac{1}{\mu_{np}},\end{aligned}$$

$$\begin{aligned}\Lambda_{12} = \Lambda_{21} &= \frac{U_{11}U_{21}}{m_n} + \frac{U_{12}U_{22}}{m_p} + \frac{U_{13}U_{23}}{m_m} \qquad (A.2)\\ &= \frac{m_n}{(m_n + m_p)m_n} - \frac{m_p}{(m_n + m_p)m_p} + 0 = 0,\end{aligned}$$

$$\begin{aligned}\Lambda_{21} &= \frac{U_{21}U_{21}}{m_n} + \frac{U_{22}U_{22}}{m_p} + \frac{U_{23}U_{23}}{m_m} \qquad (A.3)\\ &= \frac{m_n^2}{(m_n + m_p)^2 m_n} + \frac{m_p^2}{(m_n + m_p)^2 m_p} + \frac{1}{m_m}\\ &= \frac{1}{m_n + m_p} + \frac{1}{m_m} = \frac{m_n + m_p + m_m}{(m_n + m_p)m_m} = \frac{1}{\mu_{npm}}.\end{aligned}$$

Plugging this into eq. (2.16) leads to:

$$K_n + K_p + K_m - K_{cm} = -\frac{\hbar^2}{2\mu_{np}}\frac{\partial^2}{\partial \vec{r}_{np}^{\,2}} - \frac{\hbar^2}{2\mu_{npm}}\frac{\partial^2}{\partial \vec{r}_{npm}^{\,2}} \equiv K_{np} + K_{npm}. \qquad (A.4)$$

Appendix B

A matrices

In the following the A matrices are calculated for the different subsystems, where the subscript m stand for either meson. For all subsystems we have:

$$A = \sum_{i<j=1}^{N} \frac{w_{ij} w_{ij}^{\mathrm{T}}}{b_{ij}^2}. \tag{B.1}$$

For the neutron-proton subsystem we have:

$$w_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{B.2}$$

Which has to be transformed because of the change to Jacobi coordinates. The new "vector" becomes:

$$w_{12} = \begin{pmatrix} \frac{m_p}{m_n+m_p} & -\frac{m_n}{m_n+m_p} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1. \tag{B.3}$$

This means, that $A^{(d)}$ simply becomes:

$$A^{(d)} = \frac{1}{b_1^2} \tag{B.4}$$

Continuing with the subsystems with one meson, we have:

$$w_{12} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad w_{13} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad w_{23} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{B.5}$$

Transforming these leads to:

$$w_{12} = \begin{pmatrix} \frac{m_p}{m_n+m_p} & -\frac{m_n}{m_n+m_p} & 0 \\ \frac{m_m}{m_n+m_p+m_m} & \frac{m_m}{m_n+m_p+m_m} & -\frac{m_n+m_p}{m_n+m_p+m_m} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{B.6}$$

$$w_{13} = \begin{pmatrix} \frac{m_p}{m_n+m_p} & -\frac{m_n}{m_n+m_p} & 0 \\ \frac{m_m}{m_n+m_p+m_m} & \frac{m_m}{m_n+m_p+m_m} & -\frac{m_n+m_p}{m_n+m_p+m_m} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{m_p}{m_n+m_p} \\ 1 \end{pmatrix}, \tag{B.7}$$

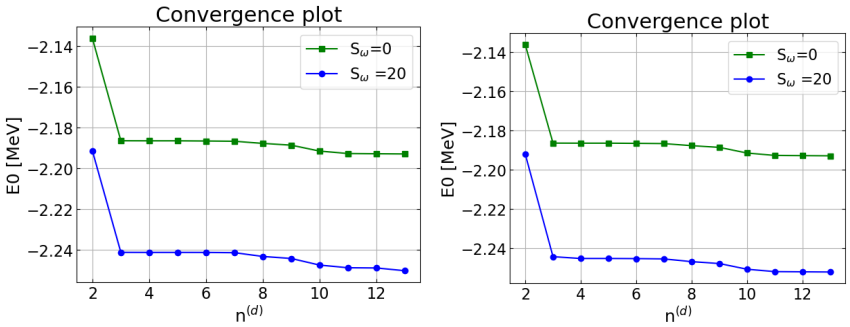
$$w_{23} = \begin{pmatrix} \frac{m_p}{m_n+m_p} & -\frac{m_n}{m_n+m_p} & 0 \\ \frac{m_m}{m_n+m_p+m_m} & \frac{m_m}{m_n+m_p+m_m} & -\frac{m_n+m_p}{m_n+m_p+m_m} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{m_n}{m_n+m_p} \\ 1 \end{pmatrix}. \tag{B.8}$$

This results in the matrices:

$$A^{(m)} = \left(\frac{1}{b_1^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{b_2^2} \begin{pmatrix} \frac{m_p^2}{(m_n+m_p)^2} & \frac{m_p}{m_n+m_p} \\ \frac{m_p}{m_n+m_p} & 0 \end{pmatrix} + \frac{1}{b_3^2} \begin{pmatrix} \frac{m_n^2}{(m_n+m_p)^2} & \frac{m_n}{m_n+m_p} \\ \frac{m_n}{m_n+m_p} & 0 \end{pmatrix} \right) \tag{B.9}$$

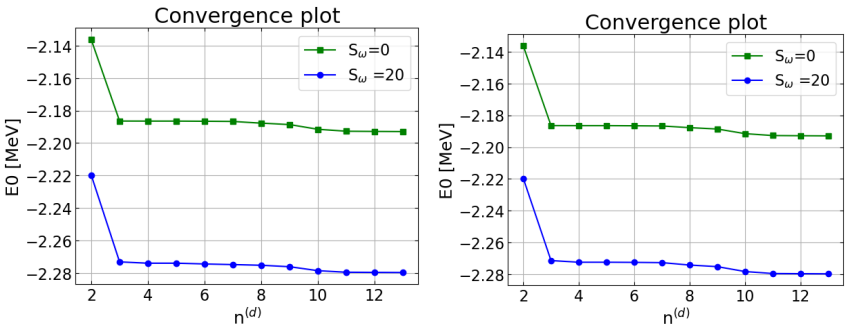
Appendix C

Figures



(a) $\omega_x = 1$ (b) $\omega_y = 1$

FIGURE C.1: The convergence of the energy for different number of Gaussian's in the np subsystem with (S_ω=20 MeV) and without (S_ω=0 MeV) the ω meson (b_ω is set to 1.4 fm). $n^{(\sigma)} = 6$ and $n^{(\omega)} = 1$ are fixed and $n^{(d)}$ is varied. In the left figure (a) $\omega_x = 1$ is used and in figure (b) $\omega_y = 1$. The simplest form of the creation operator is used.



(a) $\omega_x = 1$ (b) $\omega_y = 1$

FIGURE C.2: The convergence of the energy for different number of Gaussian's in the np subsystem with (S_ω=20 MeV) and without (S_ω=0 MeV) the ω meson (b_ω is set to 1.4 fm). $n^{(\sigma)} = 6$ and $n^{(\omega)} = 1$ are fixed and $n^{(d)}$ is varied. In the left figure (a) $\omega_x = 1$ is used and in figure (b) $\omega_y = 1$. The squared form of the creation operator is used.