

1 Ordinary least squares problem

1.1 Introduction

A system of linear equations is considered *overdetermined* if there are more equations than unknown variables. If all equations of an overdetermined system are linearly independent, the system has no exact solution.

An *ordinary least-squares problem* (also called *linear least-squares problem*) is the problem of finding an approximate solution to an overdetermined linear system. It often arises in applications where a theoretical model is fitted to experimental data.

1.2 Ordinary least-squares problem

Consider a linear system

$$\mathbf{A}\mathbf{c} = \mathbf{b}, \quad (1)$$

where \mathbf{A} is a $n \times m$ matrix, \mathbf{c} is an m -component vector of unknown variables and \mathbf{b} is an n -component vector of the right-hand side terms. If the number of equations n is larger than the number of unknowns m , the system is overdetermined and generally has no solution.

However, it is still possible to find an approximate solution — the one where $\mathbf{A}\mathbf{c}$ is only approximately equal \mathbf{b} — in the sense that the Euclidean norm of the difference between $\mathbf{A}\mathbf{c}$ and \mathbf{b} is minimized,

$$\mathbf{c} : \min_{\mathbf{c}} \|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2. \quad (2)$$

The problem (2) is called the ordinary least-squares problem and the vector \mathbf{c} that minimizes $\|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2$ is called the *least-squares solution*.

1.3 Solution via QR-decomposition

The linear least-squares problem can be solved by QR-decomposition. The matrix \mathbf{A} is factorized as $\mathbf{A} = \mathbf{Q}\mathbf{R}$, where \mathbf{Q} is $n \times m$ matrix with orthogonal columns, $\mathbf{Q}^T\mathbf{Q} = \mathbf{1}$, and \mathbf{R} is an $m \times m$ upper triangular matrix. The Euclidean norm $\|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2$ can then be rewritten as

$$\begin{aligned} \|\mathbf{A}\mathbf{c} - \mathbf{b}\|^2 &= \|\mathbf{Q}\mathbf{R}\mathbf{c} - \mathbf{b}\|^2 \\ &= \|\mathbf{R}\mathbf{c} - \mathbf{Q}^T\mathbf{b}\|^2 + \|(1 - \mathbf{Q}\mathbf{Q}^T)\mathbf{b}\|^2 \\ &\geq \|(1 - \mathbf{Q}\mathbf{Q}^T)\mathbf{b}\|^2. \end{aligned} \quad (3)$$

The term $\|(1 - \mathbf{Q}\mathbf{Q}^T)\mathbf{b}\|^2$ is independent of the variables \mathbf{c} and can not be reduced by their variations. However, the term $\|\mathbf{R}\mathbf{c} - \mathbf{Q}^T\mathbf{b}\|^2$ can be reduced down to zero by solving the $m \times m$ system of linear equations

$$\mathbf{R}\mathbf{c} = \mathbf{Q}^T\mathbf{b} . \quad (4)$$

The system is right-triangular and can be readily solved by back-substitution.

Thus the solution to the ordinary least-squares problem (2) is given by the solution of the triangular system (4).

1.4 Ordinary least-squares curve fitting

Ordinary least-squares curve fitting is a problem of fitting n (experimental) data points $\{x_i, y_i \pm \Delta y_i\}_{i=1, \dots, n}$, where Δy_i are experimental errors, by a linear combination, $F_{\mathbf{c}}$, of m functions $\{f_k(x)\}_{k=1, \dots, m}$,

$$F_{\mathbf{c}}(x) = \sum_{k=1}^m c_k f_k(x) , \quad (5)$$

where the coefficients c_k are the fitting parameters.

The objective of the least-squares fit is to minimize the square deviation, called χ^2 , between the fitting function $F_{\mathbf{c}}(x)$ and the experimental data,

$$\chi^2 = \sum_{i=1}^n \left(\frac{F(x_i) - y_i}{\Delta y_i} \right)^2 . \quad (6)$$

where the individual deviations from experimental points are weighted with their inverse errors in order to promote contributions from the more precise measurements.

Minimization of χ^2 with respect to the coefficient c_k in (5) is apparently equivalent to the least-squares problem (2) where

$$A_{ik} = \frac{f_k(x_i)}{\Delta y_i} , \quad b_i = \frac{y_i}{\Delta y_i} . \quad (7)$$

If $\mathbf{QR} = \mathbf{A}$ is the QR-decomposition of the matrix \mathbf{A} , the formal least-squares solution to the fitting problem is

$$\mathbf{c} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b} . \quad (8)$$

In practice of course one rather back-substitutes the right-triangular system $\mathbf{R}\mathbf{c} = \mathbf{Q}^T\mathbf{b}$.

1.4.1 Variances and correlations of fitting parameters

Suppose δy_i is a small deviation of the measured value of the physical observable at hand from its exact value. The corresponding deviation δc_k of the fitting coefficient is then given as

$$\delta c_k = \sum_i \frac{\partial c_k}{\partial y_i} \delta y_i . \quad (9)$$

In a good experiment the deviations δy_i are statistically independent and distributed normally with the standard deviations Δy_i . The deviations (9) are then also distributed normally with *variances*

$$\langle \delta c_k \delta c_k \rangle = \sum_i \left(\frac{\partial c_k}{\partial y_i} \Delta y_i \right)^2 = \sum_i \left(\frac{\partial c_k}{\partial b_i} \right)^2 . \quad (10)$$

The standard errors in the fitting coefficients are then given as the square roots of variances,

$$\Delta c_k = \sqrt{\langle \delta c_k \delta c_k \rangle} = \sqrt{\sum_i \left(\frac{\partial c_k}{\partial b_i} \right)^2} . \quad (11)$$

The variances are diagonal elements of the *covariance matrix*, Σ , made of *covariances*,

$$\Sigma_{kq} \equiv \langle \delta c_k \delta c_q \rangle = \sum_i \frac{\partial c_k}{\partial b_i} \frac{\partial c_q}{\partial b_i} . \quad (12)$$

Covariances $\langle \delta c_k \delta c_q \rangle$ are measures of to what extent the coefficients c_k and c_q change together if the measured values y_i are varied. The normalized covariances,

$$\frac{\langle \delta c_k \delta c_q \rangle}{\sqrt{\langle \delta c_k \delta c_k \rangle \langle \delta c_q \delta c_q \rangle}} \quad (13)$$

are called *correlations*.

Using (12) and (8) the covariance matrix can be calculated as

$$\Sigma = \left(\frac{\partial \mathbf{c}}{\partial \mathbf{b}} \right) \left(\frac{\partial \mathbf{c}}{\partial \mathbf{b}} \right)^\top = R^{-1} (R^{-1})^\top = (R^\top R)^{-1} = (A^\top A)^{-1} . \quad (14)$$

The square roots of the diagonal elements of this matrix provide the estimates of the errors $\Delta \mathbf{c}$ of the fitting coefficients,

$$\Delta c_k = \sqrt{\Sigma_{kk}} \Big|_{k=1\dots m} , \quad (15)$$

and the (normalized) off-diagonal elements provide the estimates of their correlations.

Here is a Python implementation of the ordinary least squares fit via QR decomposition,

```
def lsfit ( fs:list , x:vector , y:vector , dy:vector ) :
    n = x.size ; m = len( fs )
    A = matrix( n,m )
    b = vector( n )
    for i in range( n ):
        b[ i ] = y[ i ] / dy[ i ]
        for k in range( m ): A[ i , k ] = fs[ k ]( x[ i ] ) / dy[ i ]
    ( Q,R ) = gramschmidt. qr( A )
    c = R. back_substitute( Q.T() * b )
    inverse_R = R. back_substitute( matrix. id_matrix( m ) )
    S = inverse_R * inverse_R.T()
    return ( c , S )
```

An illustration of a fit is shown on Figure 1 where a polynomial is fitted to a set of data.

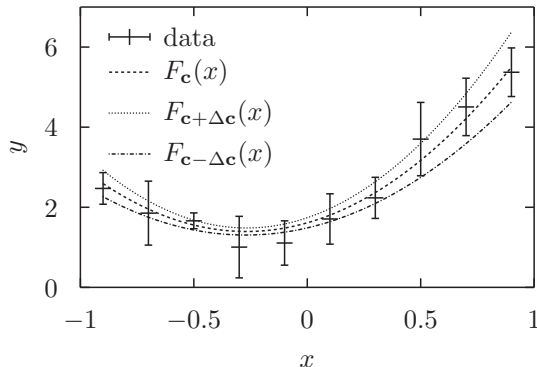


Figure 1: Ordinary least squares fit of $F_{\mathbf{c}}(x) = c_1 + c_2x + c_3x^2$ to a set of data. Shown are fits with optimal coefficients \mathbf{c} as well as with $\mathbf{c} + \Delta\mathbf{c}$ and $\mathbf{c} - \Delta\mathbf{c}$.

1.5 Singular value decomposition

Under the *thin singular value decomposition* we shall understand a representation of a tall $n \times m$ ($n > m$) matrix \mathbf{A} in the form

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\top} , \quad (16)$$

where \mathbf{U} is an orthogonal $n \times m$ matrix ($\mathbf{U}^T \mathbf{U} = \mathbf{1}$), \mathbf{S} is a square $m \times m$ diagonal matrix with non-negative real numbers on the diagonal (called singular values of matrix \mathbf{A}), and \mathbf{V} is a square $m \times m$ orthogonal matrix ($\mathbf{V}^T \mathbf{V} = \mathbf{1}$).

Singular value decomposition can be used to solve our linear least squares problem $\mathbf{A}\mathbf{c} = \mathbf{b}$. Indeed inserting the decomposition into the equation gives

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{c} = \mathbf{b} . \quad (17)$$

Multiplying from the left with \mathbf{U}^T and using the orthogonality of \mathbf{U} one gets the projected equation

$$\mathbf{S}\mathbf{V}^T \mathbf{c} = \mathbf{U}^T \mathbf{b} . \quad (18)$$

This is a square system which can be easily solved first by solving the diagonal system

$$\mathbf{S}\mathbf{y} = \mathbf{U}^T \mathbf{b} \quad (19)$$

for \mathbf{y} and then obtaining \mathbf{c} as

$$\mathbf{c} = \mathbf{V}\mathbf{y} . \quad (20)$$

The covariance matrix (14) can be calculated as

$$\mathbf{\Sigma} = (\mathbf{A}^T \mathbf{A})^{-1} = (\mathbf{V}\mathbf{S}^2\mathbf{V}^T)^{-1} = \mathbf{V}\mathbf{S}^{-2}\mathbf{V}^T . \quad (21)$$

Singular value decomposition can be found by diagonalising the $m \times m$ symmetric positive semi-definite matrix $\mathbf{A}^T \mathbf{A}$ (although this method is not the best for practical calculations, it would do as an educational tool),

$$\mathbf{A}^T \mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T , \quad (22)$$

where \mathbf{D} is a diagonal matrix with eigenvalues of the matrix $\mathbf{A}^T \mathbf{A}$ on the diagonal and \mathbf{V} is the matrix of the corresponding eigenvectors. Indeed it is easy to check that the sought decomposition can be constructed as $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$ where $\mathbf{S} = \mathbf{D}^{1/2}$, $\mathbf{U} = \mathbf{A}\mathbf{V}\mathbf{D}^{-1/2}$.