Action in general relativity

The equation for the gravitational field should be derived from the action, S, in the form

$$S = S_M + S_G \,, \tag{1}$$

where S_G is the action of the gravitational field, to be determined later, and S_M is the action of matter,[1] to be taken—in a generally covariant form—directly from special relativity.

Unlike electromagnetism, where there is an explicit coupling term between the electromagnetic field and charges, $-e\int A_a dx^a$, in general relativity there is no explicit coupling terms between gravitation and matter. Indeed, in general relativity gravitation is not a matter field—like the electromagnetic field—but rather the geometry of space-time. Therefore the action of the matter in general relativity has the same form as in special relativity, only written in a generally covariant form. The matter then couples to gravitation through the metric tensor in the action of the matter.

Since the gravitational field is contained in the metric tensor of the time-space, g_{ab}, the variational principle to determine the gravitational field created by a given distribution of matter should be formulated as

> *The actual gravitational field, g_{ab}, is that for which a small variation of the field, $g_{ab} + \delta g_{ab}$, leads to vanishing variation of the action,*
>
> $$\delta S\Big|_{g_{ab}\to g_{ab}+\delta g_{ab}} = 0 \,. \tag{2}$$

The source of the gravitational field is then determined by the variation of the matter action over the metric tensor. We shall show that it is given by the stress-energy-momentum tensor of matter. Indeed in analogy to electromagnetism—where the sources of the electromagnetic field are the charged currents—the source of the gravitational field must be the stress-energy-momentum tensor of matter.

Action of matter in general relativity

The action of matter in general relativity has the same form as in special relativity, only rewritten, if needed, in a generally covariant way. Particularly, one has to substitute

$$d\Omega \;\to\; \sqrt{-g}\,d\Omega \,, \tag{3}$$

$$\partial^a\varphi \equiv \varphi^{,a} \;\to\; g^{ab}\partial_b\varphi \equiv g^{ab}\varphi_{,b} \,, \tag{4}$$

$$\partial_a A^b \equiv A^b_{,a} \;\to\; D_a A^b \equiv A^b_{;a} \,. \tag{5}$$

For example, the action of the electromagnetic field becomes

$$-\frac{1}{16\pi}\int F^{ab}F_{ab}\sqrt{-g}\,d\Omega \,, \tag{6}$$

the coupling between the electromagnetic field and currents,

$$-\int A_a j^a \sqrt{-g}\,d\Omega \,, \tag{7}$$

the action of a hypothetical massless scalar field ϕ,

$$-\int g^{ab}\partial_a\varphi\partial_b\varphi\sqrt{-g}\,d\Omega \,. \tag{8}$$

[1] "Matter" in general relativity is everything but the gravitational field (the latter being curvature of space-time).

Metric variation of matter action and energy-momentum tensor

We shal show here that the variation of the matter action under variation of the metric tensor produces the stress-energy-momentum tensor of matter. The latter becomes the source of the gravitational field.

The action of matter is typically written in terms of the Lagrangian of the matter, $\mathcal{L}$,

$$S_M = \int \mathcal{L}\sqrt{-g}d\Omega\,. \tag{9}$$

The variation of this action under a variation $g^{ab} \to g^{ab} + \delta g^{ab}$ of the metric tensor can be written in terms of a symmetric tensor T_{ab},

$$\delta S_M \doteq \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega = -\frac{1}{2}\int T^{ab}\delta g_{ab}\sqrt{-g}d\Omega,$$

where[2]

$$\frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab} \doteq \delta(\sqrt{-g}\mathcal{L})\,. \tag{10}$$

The tensor T_{ab} must be the stress-energy-momentum tensor, since in a flat space it satisfies the conservation law $T^{ab}_{,b} = 0$. Indeed, consider an infinitesimal coordinate transformation,

$$x^a \to x'^a = x^a + \epsilon^a\,. \tag{11}$$

The variation of the metric tensor under this transformation can be written as[3]

$$\delta g^{ab} = \epsilon^{a;b} + \epsilon^{b;a}\ ,\ \delta g_{ab} = -\epsilon_{a;b} - \epsilon_{b;a}\,. \tag{16}$$

The variation of the action then takes the form

$$\delta S = \int T_{ab}\epsilon^{a;b}\sqrt{-g}d\Omega\,. \tag{17}$$

Integrating by parts[4] gives

$$\delta S = -\int T^b_{a;b}\epsilon^a\sqrt{-g}d\Omega\,. \tag{18}$$

The variation of the action under coordinate transformation is zero. Thus the tensor T^a_b satisfies the equation

$$T^{ab}_{;b} = 0\,, \tag{19}$$

which in a flat space turns into the stress-energy-momentum conservation equation $T^{ab}_{,b} = 0$. One can thus assume that the tensor is proportional to the canonical stress-energy-momentum tensor. Direct calculations show that the proportionality factor is equal unity.

[2]From $g_{ab}g^{bc} = \delta^c_a$ follows $g_{ab}\delta g^{bc} = -\delta g_{ab}g^{bc}$ and therefore $T_{ab}\delta g^{ab} = -T^{ab}\delta g_{ab}$.

[3]Differentiating $x'^a = x^a + \epsilon^a$ gives

$$x'^a_{,b} = \delta^a_b + \epsilon^a_{,b}\,. \tag{12}$$

Substituting this into the transformation rule for the metric tensor,

$$g_{ab}(x) = x'^c_{,a}x'^d_{,b}g'_{cd}(x')\,, \tag{13}$$

gives

$$\begin{aligned} g_{ab}(x) &= (\delta^c_a + \epsilon^c_{,a})(\delta^d_b + \epsilon^d_{,b})\left(g'_{cd}(x) + g_{cd,e}\epsilon^e + O((\epsilon)^2)\right) && (14)\\ &= g'_{ab} + g_{ad}\epsilon^d_{,b} + g_{bd}\epsilon^d_{,a} + g_{ab,e}\epsilon^e + O\left((\epsilon)^2\right) = g'_{ab} + \epsilon_{a;b} + \epsilon_{b;a} + O\left((\epsilon)^2\right)\,. && (15)\end{aligned}$$

[4]using $(AB)_{;a} = A_{;a}B + AB_{;a}$ and $\sqrt{-g}A^c_{;c} = (\sqrt{-g}A^c)_{,c}$ and that a total differential does not contribute to variation.

Exercises

1. The Lagrangian density of the electromagnetic field is given as
$$\mathcal{L} = -\frac{1}{16\pi} F_{ab}F^{ab}.$$
Calculate the corresponding stress-energy-momentum tensor, T_{ab}, using the "metric" definition,
$$\delta\left(\sqrt{-g}\mathcal{L}\right)_{g^{ab}\to g^{ab}+\delta g^{ab}} = \frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab}\,.$$
Answer:
$$T_{ab} = \frac{1}{16\pi}\left(F_{cd}F^{cd}g_{ab} - 4F_{ac}F_b{}^c\right)\,.$$

2. Consider a scalar field φ whose action in special relativity is given as
$$S = \int d\Omega\left(\frac{1}{2}\varphi^{,a}\varphi_{,a} - \frac{1}{2}m^2\varphi^2\right)\,.$$
   (a) Calculate its "translation-invariance" stress-energy-momentum tensor,
   $$T^a_b = \frac{\partial\mathcal{L}}{\partial\varphi_{,a}}\varphi_{,b} - \mathcal{L}\delta^a_b\,.$$
   (b) Rewrite the action in a generally covariant form and calculate its "metric" stress-energy-momentum tensor,
   $$\delta\left(\sqrt{-g}\mathcal{L}\right)_{g^{ab}\to g^{ab}+\delta g^{ab}} = \frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab}$$
   Discuss the results.

3. Show that for an infinitesimally small coordinate transformation $x^a \to x^a + \epsilon^a$ the variation of the metric tensor is given as
$$\delta g^{ab} = \epsilon^{a;b} + \epsilon^{b;a}\ ,\ \delta g_{ab} = -\epsilon_{a;b} - \epsilon_{b;a}\,.$$

4. Derive—by metric variation—the stress-energy-momentum tensor of a particle with mass m and argue that it is the correct tensor.
Hints:
   (a) The action is $S = -m\int ds$.
   (b) Its variation over δg_{ab} must look like $\delta S = -\frac{1}{2}\int ds T^{ab}\delta g_{ab}\,.$

5. Argue that the equation $j^a_{,a} = 0$ represents a conservation law. What about $j^a_{;a} = 0$?

6. Argue that the equation $T^{ab}_{,a} = 0$ represents a conservation law. What about $T^{ab}_{;a} = 0$?

7. Prove eq. (18), that is, that
$$\int T^{ab}\epsilon_{a;b}\sqrt{-g}d\Omega = -\int T^{ab}_{;b}\epsilon_a\sqrt{-g}d\Omega\,.$$

8. Argue that under an arbitrary infinitesimal coordinate transformation $x^a \to x^a + \epsilon^a$ the variation of the action of the matter is given as
$$0 = \delta S_m = -\frac{1}{2}\int T^{ab}\delta g_{ab}\sqrt{-g}d\Omega = \int T^{ab}_{;b}\epsilon_a\sqrt{-g}d\Omega\,.$$
Does this imply that i) $T^{ab} = 0$? ii) $T^{ab}_{;b} = 0$?