Schwarzschild solution

Schwarzschild solution is the spherically symmetric static vacuum solution of the Einstein equation. It describes the gravitational field outside a slowly rotating spherical body, like a star, a planet, or a black hole. Karl Schwarzschild has found this solution in 1915 and published it in January 1916 shortly after the publication of Einstein's general theory of relativity.

Schwarzschild metric

Schwarzschild solution is a static spherically symmetric solution of the vacuum Einstein equation

$$R_{ab} = 0\,. \tag{1}$$

The spherically symmetric static metric can be generally assumed to have the following form,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{2}$$

where A and B are some yet unknown functions of radius r.

Now we have to calculate the Ricci tensor for this metric and and put it into the vacuum equation above. That will give us the differential equations for A and B. Instead of duing this tedious work manually, one can use the following GNU Maxima script,

```
derivabbrev:true; /* a better notation for derivatives */
load(ctensor); /* load the package to deal with tensors */
ct_coords:[t,r,o,p]; /* our coordiantes: t r theta phi */
depends([A,B],[r]); /* the functions A and B depend on r */
lg:ident(4); /* set up a 4x4 identity matrix */
lg[1,1]: A; /* g_{tt} = A */
lg[2,2]:-B; /* g_{rr} = -B */
lg[3,3]:-r^2; /* g_{\theta\theta} */
lg[4,4]:-r^2*sin(o)^2; /* g_{\phi\phi} */
cmetric(); /* compute the prerequisites for further calculations */
christof(mcs); /* calculates Christoffel symbols, mcs[b,c,a]=\Gamma^a_{bc} */
ricci(true); /* calculates ric[a,b], the covariant symmetric Ricci tensor */
```

which calculates analytically the Christoffel symbols[1] and the Ricci tensor[2] for this metric. The vacuum Einstein equation, $R_{ab} = 0$, can then be integrated analytically[3] which gives the famous Schwarzschild metric,

$$ds^2 = \left(1 - \frac{r_g}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{3}$$

The integration constant r_g is determined from the Newtonian limit[4], $r_g = 2GM/c^2$, where M is the mass of the central body ($r_g = 2M$ in the units $G = c = 1$). It is called gravitational or Schwarzschild radius. The gravitational radius for the Earth is about 9mm, for the Sun – about 3km.

[1] $\Gamma^r_{rr} = \frac{1}{2}\frac{B'}{B}\,,\ \Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A}\,,\ \Gamma^r_{tt} = \frac{1}{2}\frac{A'}{B}\,,\ \Gamma^\theta_{\theta r} = \frac{1}{r}\,,\ \Gamma^r_{\theta\theta} = -\frac{r}{B}\,,\ \Gamma^\phi_{\phi r} = \frac{1}{r}\,,\ \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B}\,,\ \Gamma^\phi_{\phi\theta} = \cot\theta\,,\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,.$

[2]
$$R_{tt} = \frac{A''}{2B} + \frac{A'}{B}\left(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A}\right)\,,$$
$$R_{\theta\theta} = 1 - \left(\frac{r}{B}\right)' - \frac{1}{2}\left(\frac{A'}{A} + \frac{B'}{B}\right)\frac{r}{B}\,,$$
$$R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}\,.$$

[3] Making a linear combination $BR_{tt} + AR_{rr} = 0$ gives $A'B + AB' = 0 \Rightarrow AB = 1 \Rightarrow \frac{A'}{A} + \frac{B'}{B} = 0$. Then $R_{\theta\theta} = 0$ gives $B = \frac{1}{1-\frac{r_g}{r}}$, $A = 1 - \frac{r_g}{r}$, where r_g is an integration constant.

[4] $g_{00} \xrightarrow{r\to\infty} 1 + 2\phi = 1 - 2\frac{GM}{c^2 r}$

Motion in the Schwarzschild metric

Massive bodies

Massive bodies move along geodesics, described by the geodesic equation

$$\frac{d}{ds}\left(g_{ab}u^b\right) = \frac{1}{2}g_{bc,a}u^b u^c \,. \tag{4}$$

For $a = t, \theta, \phi$ the corresponding equations in the Schwarzschild metric (3) are

$$\frac{d}{ds}\left[\left(1 - \frac{2M}{r}\right)\frac{dt}{ds}\right] = 0 \,, \tag{5}$$

$$\frac{d}{ds}\left[r^2\frac{d\theta}{ds}\right] = r^2 \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 \,, \tag{6}$$

$$\frac{d}{ds}\left[r^2\sin^2\theta\frac{d\phi}{ds}\right] = 0 \,. \tag{7}$$

The r-equation can be conveniently obtained by dividing the Schwarzschild metric (3) by ds^2,

$$1 = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right] . \tag{8}$$

Considering equatorial motion, the first three equations can be integrated as

$$\theta = \frac{\pi}{2} \,, \quad r^2\frac{d\phi}{ds} = J \,, \quad \left(1 - \frac{2M}{r}\right)\frac{dt}{ds} = E \,, \tag{9}$$

where J and E are constants. The fourth equation then becomes

$$1 = \frac{E^2}{1 - \frac{2M}{r}} - \frac{\frac{J^2}{r^4}r'^2}{1 - \frac{2M}{r}} - \frac{J^2}{r^2} \,, \tag{10}$$

where $r' \equiv \frac{dr}{d\phi}$. Traditionally one makes a variable substitution $r = 1/u$,

$$(1 - 2Mu) = E^2 - J^2u'^2 - J^2u^2(1 - 2Mu), \tag{11}$$

and then differentiates the equation once. This gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2 \,. \tag{12}$$

In this form the last term is a relativistic correction to the otherwise non-relativistic equation.

Light rays.

The rays of light travel along the null-geodesics where $ds^2 = 0$. Consequently instead of ds one needs to use some parameter $d\lambda$ in the geodesic equations $\frac{Dk^a}{d\lambda} = 0$, where $k^a = \frac{dx^a}{d\lambda}$ and also the unity in the left-hand side of equation (8) has to be substituted with zero. This immediately leads to the equation

$$u'' + u = 3Mu^2 \,, \tag{13}$$

which describes the trajectory of a ray of light in the Schwarzschild metric.

In the absence of the central body, $M = 0$ the space becomes flat, and equation (13) turns into equation for a straight line.

Exercises

1. Consider a non-relativistic (Newtonian) equatorial ($\theta = \pi/2$) motion of a planet with mass m around a star with mass M described by a Lagrangian[5]
$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mM}{r}\,,$$
Write down the Euler-Lagrange equations,
$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}\,,$$
for $q = r$ and ϕ. Using the first integral $r^2\dot{\phi} = J$ rewrite the r-equation as an equation for the function $u(\phi)$, where $u = 1/r$, and compare with (12).

2. Show that in Newtonian mechanics an equatorial ($\theta = \pi/2$) trajectory of a light ray is described by the equation
$$u'' + u = 0\,,$$
where $u \doteq \frac{1}{r}$ and $u' \doteq \frac{du}{d\phi}$.

3. From the expression for the Schwarzschild metric calculate the Christoffel symbols $\Gamma^r_{\phi\phi}$, $\Gamma^\phi_{r\phi}$.

4. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. Answer: $r = \frac{3}{2}(2M)$.

5. Show, that in the Newtonian limit, $g_{00} = 1 + 2\phi/c^2$, the geodesic equation,
$$\frac{du^a}{ds} = -\Gamma^a_{bc}u^b u^c\,,$$
is consistent with the Newton's equation of motion,
$$\ddot{\vec{r}} = -\vec{\nabla}\phi\,.$$

6. Show that for the equatorial orbit in the Schwarzschild metric the quantity u_ϕ is conserved (where u^ϕ is the ϕ-component of the four-velocity $u^a \doteq dx^a/ds$).

7. Build a Maxima script that proves (analytically) that the Schwarzschild metric is indeed a solution to the vacuum Einstein equation. Hint: `factor(ric[1,1]);`.

[5]where the dot denotes the temporal derivative, $\dot{r} \equiv \frac{dr}{dt}$.