Three nucleon scattering in a "three dimensional" approach

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PARTIAL WAVES - EXAMPLE

- For the moment let's focus on the 2N system.
- Let's try to calculate the 2N transition operator.

PARTIAL WAVES - SYMMETRIZATION

$|(00)0\rangle$ $|(11)0\rangle$ $|(01)1\rangle$ $|(10)1\rangle$ $|(11)1\rangle$ $|(21)1\rangle$ $|(11)2\rangle$ $|(20)2\rangle$ $|(21)2\rangle$ $|(31)2\rangle$ $|(21)3\rangle$ $|(30)3\rangle$ $|(31)3\rangle$ $|(41)3\rangle$ $\cdots$

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PARTIAL WAVES - SYMMETRIZATION

- $\check{V}$ is the 2N potential.
- Each $\blacksquare$ lives in a subspace with given orbital angular momentum l, spin s and total angular momentum j and different momentum magnitude states:

$$\langle |\mathbf{p}'|(l's')j' \mid \ldots \mid |\mathbf{p}|(ls)j \rangle$$

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$\|(00)0\rangle$	■			■				■				■			$\cdots$
$\|(11)0\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(01)1\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(10)1\rangle$	■			■				■				■			$\cdots$
$\|(11)1\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(21)1\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(11)2\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(20)2\rangle$	■			■				■				■			$\cdots$
$\|(21)2\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(31)2\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(21)3\rangle$		■	■		■	■	■		■	■	■		■	■	$\cdots$
$\|(30)3\rangle$	■			■				■				■			$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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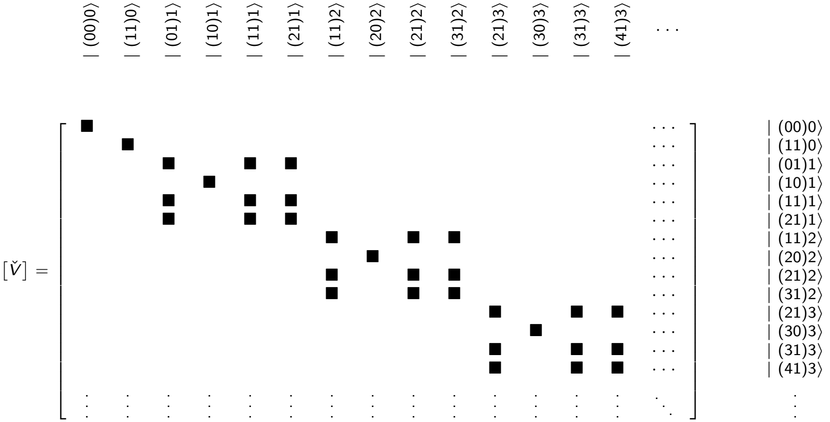

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- Perform PWD on each operator of, eg., LSE $\check{t} = \check{V} + \check{V}\check{G}_0\check{t}$.
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PARTIAL WAVES - PROS AND CONS

- +
  - Battle tested.
  - Small numerical workload.
- −
  - Implementation requires heavily oscilating functions.
  - It is not always obvious how many partial waves need to be taken into account.
  - This is more complicated for three or more particles and different coupling schemes.
  - Convergence problems for higher energies.

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3D - EXAMPLE

- Lets take the 2N transition operator.
- Assume we are working in momentum space with $\mathbf{p}' = (p'_x, p'_y, p'_z)$ being the final and $\mathbf{p} = (p_x, p_y, p_z)$ being the initial momentum of the two nucleons.

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3D - SIZE OF THE PROBLEM

- We would like to calculate the full transition operator. This is equivalent to calculating, for every $\mathbf{p}'$ and every $\mathbf{p}$, the matrix element $\langle \mathbf{p}' \mid \check{t} \mid \mathbf{p} \rangle$.
- This matrix element is an operator in spin space and has the form:

$$\left[\langle \mathbf{p}' \mid \check{t} \mid \mathbf{p} \rangle \right] =$$

$$\begin{bmatrix} t_{11}(p'_x, p'_y, p'_z, p_x, p_y, p_z) & t_{12}(\ldots) & t_{13}(\ldots) & t_{14}(\ldots) \\ t_{21}(\ldots) & t_{22}(\ldots) & t_{23}(\ldots) & t_{24}(\ldots) \\ t_{31}(\ldots) & t_{32}(\ldots) & t_{33}(\ldots) & t_{34}(\ldots) \\ t_{41}(\ldots) & t_{42}(\ldots) & t_{43}(\ldots) & t_{44}(\ldots) \end{bmatrix} .$$

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- We need to calculate 16 functions of 6 real parameters that satisfy the LSE.
- We know that the solution has to satisfy appropriate symmetries.
- Can we use this to simplify the problem?

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3D - SYMMETRIZATION

- The general operator form of the two nucleon potential and transition operator is well known [Phys. Rev. 96 1654 (1954)].
- The matrix element in momentum space can be written as a linear combination of 6 scalar functions t_i and spin operators $[w_i]$:

$$\left[\langle \mathbf{p}' \mid \check{t} \mid \mathbf{p} \rangle\right] = \sum_{i=1}^{6} t_i(|\mathbf{p}'|, |\mathbf{p}|, \hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \left[\check{w}_i(\mathbf{p}, \mathbf{p})\right].$$

- Instead of calculating 16 functions of 6 real variables we now only need to calculate 6 functions of 3 variables.
- Couple orders of magnitude less numerical work!

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- If $\check{R}$ is a spatial rotation, we require that operator $\check{X}$:

$$\check{R}^{-1}\check{X}\check{R} = \check{X}.$$

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Here x is a scalar function of momenta and O is an operator in spin space and … are momenta.

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$$(\check{\mathbf{v}}_1 \times (\check{\mathbf{v}}_2 \times \check{\mathbf{v}}_3)) \cdot (\check{\mathbf{v}}_4 \times (\check{\mathbf{v}}_5 \times (\check{\mathbf{v}}_6 \times \check{\mathbf{v}}_7)))$$

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$$\mathbb{V} = \{\check{1}, \check{\mathbf{v}}_i \cdot \check{\mathbf{v}}_j \;,\; (\check{\mathbf{v}}_i \times \check{\mathbf{v}}_j) \cdot \check{\mathbf{v}}_k\}.$$

- For example, from the previous slide, we have the following CHAINS of operators of length 3:

$$\begin{aligned}
[\check{C}_1] &= (\check{\mathbf{v}}_1 \cdot \check{\mathbf{v}}_3)(\check{\mathbf{v}}_2 \cdot \check{\mathbf{v}}_5)(\check{\mathbf{v}}_4 \times \check{\mathbf{v}}_6 \cdot \check{\mathbf{v}}_7),\\
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- In principle we have to include CHAINS of operators constructed from $\mathbb{V}$ with any number of links?

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- Not all chains are unique. $\left[\check{C}_i\right]$ not unique if $\left[\check{C}_i\right] = \sum_{j \neq i} c_j \left[\check{C}_j\right]$.
- The algorithm in [*Eur. Phys. J.* A 52:188 (2016)] can be summarized:
  - Start with all chains of length 1.
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  - At some point adding further links to the chains does not add any new unique operators.
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&\check{\mathbf{p}} \cdot \check{\sigma}(2) \\
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- We can use a simple symmetrization procedure.
- Let $\mathbb{D}$ be a grup of transformations constructed from **parity, time reversal, Hermitian conjugate and two particle exchange**.
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$$\check{T} = \check{t}\check{P} + \check{t}\check{G}_0\check{P}\check{T}.$$
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- Compare 3D approach and “battle tested” PWD approach.

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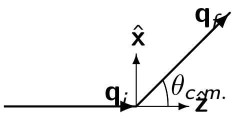

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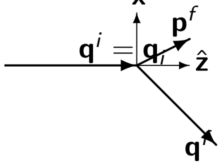

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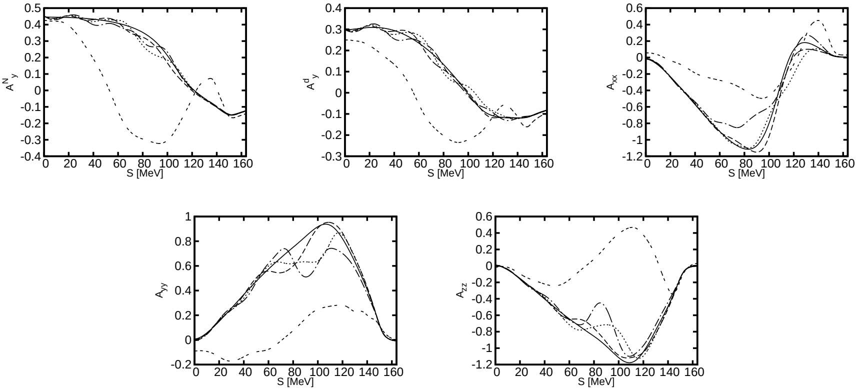

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- First order results for neutron deuteron scattering suggest that the 3D approach can be used to achieve convergence at higher energies.
- There is a necessity to construct new general operator forms.
- Constructing $\check{T} \mid \phi\rangle$ can lead to efficient calculations.
- Possibility to add relativistic corrections to the calculations.

THANK YOU

The project was financed from the resources of the National Science Center (Poland) under grants No. DEC-2013/11/N/ST2/03733 and DEC-2013/10/M/ST2/00420. Some of the numerical work was performed at JSC Jülich.