# Three-body wave functions in the continuum: Application to the Coulomb case

## Goal:

- ✓ Method that permits to compute **continuum three-body wave functions** without knowledge of the asymptotic form of the wave function.
  - ➢ Therefore suitable for systems involving the Coulomb interaction.
- ✓ The method provides automatically the wave functions for **all the possible incoming channels**.
- ✓ In particular it can be used to obtain the regular three-body Coulomb functions.
- ✓ From the regular Coulomb function the corresponding irregular partner can be extracted.
- ✓ The regular and irregular Coulomb functions permit to extract the S-matrix of the process.
- ✓ The three-alpha system is taken as illustration


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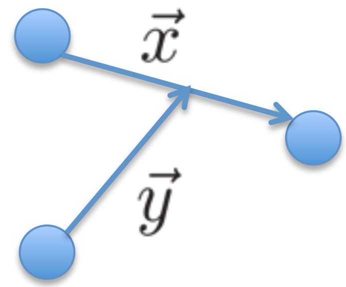
## Three-body wave functions

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2}+\frac{5}{\rho}\frac{\partial}{\partial\rho}\right)+\frac{\hbar^2}{2m\rho^2}\left(\hat{\Lambda}^2+\frac{2m\rho^2}{\hbar^2}\sum_{i<j}V_{ik}(r_{ij})\right)\right]\Psi=E\Psi$$

Typically three-body wave functions are expanded in terms of some basis containing the whole dependence on the hyperangles




$$\Omega_x \equiv \{\theta_x, \varphi_x\}$$
$$\Omega_y \equiv \{\theta_y, \varphi_y\}$$
$$\tan\alpha = x/y$$
$$\rho^2 = x^2 + y^2$$

### Bound states

**HH expansion**

$$\Psi = \frac{1}{\rho^{5/2}}\sum_q \chi_q(\rho)\mathcal{Y}_q(\Omega)$$

$$q \equiv \{K, \ell_x, \ell_y, L, s_x, S\}$$

$$\chi_q(\rho) \to e^{-\kappa\rho}$$

**Adiabatic expansion**

$$\Psi = \frac{1}{\rho^{5/2}}\sum_n f_n(\rho)\Phi_n(\rho,\Omega)$$

$$\mathcal{H}_\Omega\Phi_n(\rho,\Omega) = \lambda_n(\rho)\Phi_n(\rho,\Omega)$$

$$f_n(\rho) \to e^{-\kappa\rho}$$

The exponentially fall off asymptotic determines the energy of the bound state and its structure.

## Three-body wave functions

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2}+\frac{5}{\rho}\frac{\partial}{\partial\rho}\right)+\frac{\hbar^2}{2m\rho^2}\left(\hat{\Lambda}^2+\frac{2m\rho^2}{\hbar^2}\sum_{i<j}V_{ik}(r_{ij})\right)\right]\Psi=E\Psi$$

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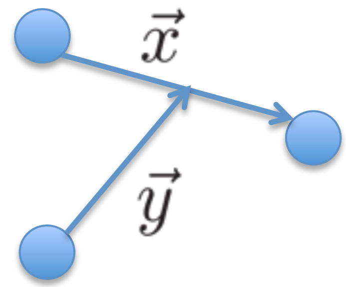


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## Continuum states

### HH expansion

$$\Psi = \frac{1}{\rho^{5/2}}\sum_q \chi_q(\rho)\mathcal{Y}_q(\Omega)$$

$$\chi_q(\rho) \to S_q(E) H^{(1)}_{K+2}(\kappa\rho)$$

$$\chi_{\tilde{q}}(\rho) \to H^{(1)}_{\tilde{K}+2}(\kappa\rho) + S_{\tilde{q}}(E) H^{(1)}_{\tilde{K}+2}(\kappa\rho)$$

### Adiabatic expansion

$$\Psi = \frac{1}{\rho^{5/2}}\sum_n f_n(\rho)\Phi_n(\rho,\Omega)$$

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The asymptotic behaviour of the continuum wave functions is not unique.

## Three-body wave functions

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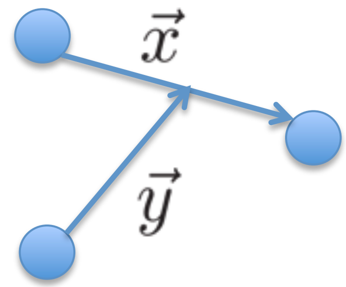


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## Continuum states

**HH expansion**

$$\Psi_{\tilde{q}} = \frac{1}{\rho^{5/2}}\sum_q \chi_{q\tilde{q}}(\rho)\mathcal{Y}_q(\Omega)$$

$$\chi_{q\tilde{q}}(\rho) \to H^{(1)}_{K+2}(\kappa\rho)\delta_{q\tilde{q}} + S_{q\tilde{q}}(E)H^{(1)}_{K+2}(\kappa\rho)$$

**Adiabatic expansion**

$$\Psi_{\tilde{n}} = \frac{1}{\rho^{5/2}}\sum_n f_{n\tilde{n}}(\rho)\Phi_n(\rho,\Omega)$$

$$f_{n\tilde{n}}(\rho) \to H^{(1)}_{K+2}(\kappa\rho)\delta_{n\tilde{n}} + S_{n\tilde{n}}(E)H^{(1)}_{K+2}(\kappa\rho)$$

The three-body problem in the continuum is equivalent to a multichannel problem

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \vdots\\ \Psi_N\end{pmatrix}$$

*N* = number of channels included in the calculation

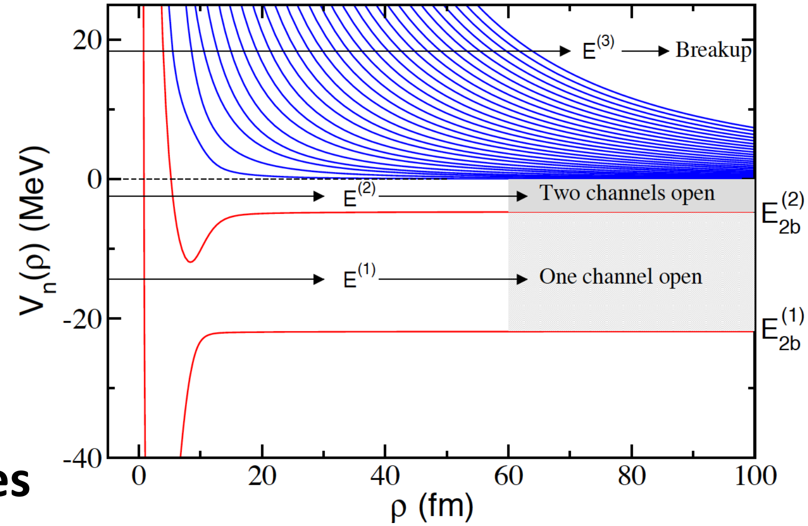
Adiabatic expansion

$$\left[-\frac{d^2}{d\rho^2}+\frac{2m}{\hbar^2}(V_n(\rho)-E)\right]f_{ni}(\rho)+\sum_{n'}\left(-2P_{nn'}\frac{d}{d\rho}-Q_{nn'}\right)f_{n'i}(\rho)=0$$

**adiabatic potentials**

For three-body continuum states convergence is typically reached after inclusion of about 20-30 adiabatic terms in the expansion

E.G., C. Romero-Redondo, A. Kievsky, M. Viviani , PRA 86 (2012) 052709




## Continuum states

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$$\Psi_{\tilde{q}}=\frac{1}{\rho^{5/2}}\sum_q \chi_{q\tilde{q}}(\rho)\mathcal{Y}_q(\Omega)$$

$$\chi_{q\tilde{q}}(\rho)\to H^{(1)}_{K+2}(\kappa\rho)\delta_{q\tilde{q}}+S_{q\tilde{q}}(E)H^{(1)}_{K+2}(\kappa\rho)$$

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# Adiabatic expansion

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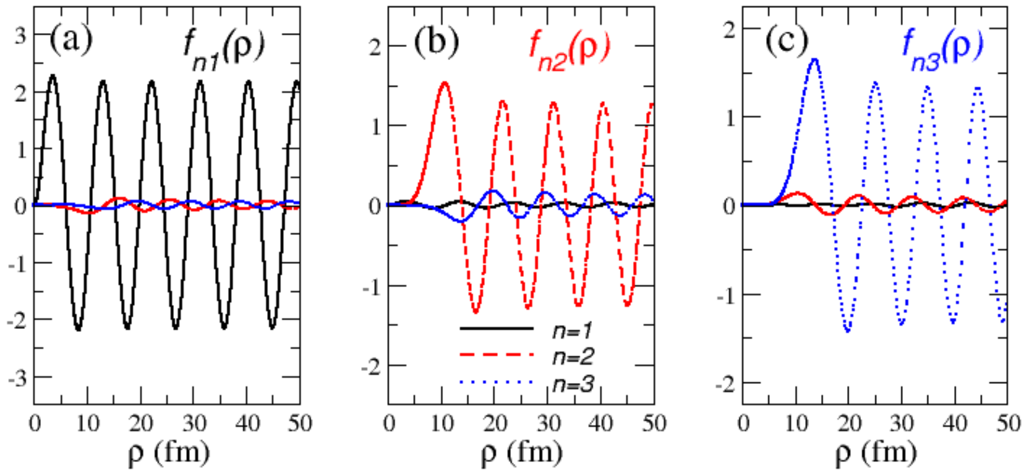


Radial matrix for the 0⁺ state of three spinless identical bosons ($Z$=0)

$E$=10 MeV

$Z$=0 → **Short-range interactions**

Adiabatic expansion

$$\Psi_{\tilde{n}}=\frac{1}{\rho^{5/2}}\sum_n f_{n\tilde{n}}(\rho)\Phi_n(\rho,\Omega)$$

$$f_{n\tilde{n}}(\rho)\to H^{(1)}_{K+2}(\kappa\rho)\delta_{n\tilde{n}}+S_{n\tilde{n}}(E)H^{(1)}_{K+2}(\kappa\rho)$$

$$\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\\ \vdots\\ \Psi_N\end{pmatrix}=\frac{1}{\rho^{5/2}}\begin{pmatrix}f_{11} & f_{21} & \dots & f_{N1}\\ f_{12} & f_{22} & \dots & f_{N2}\\ \vdots & \vdots & \vdots & \vdots\\ f_{1N} & f_{2N} & \dots & f_{NN}\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_N\end{pmatrix}=\frac{1}{\rho^{5/2}}R\Phi$$

Box boundary condition

$$\left[-\frac{d^2}{d\rho^2}+\frac{2m}{\hbar^2}(V_n(\rho)-E)\right]f_{ni}(\rho)+\sum_{n'}\left(-2P_{nn'}\frac{d}{d\rho}-Q_{nn'}\right)f_{n'i}(\rho)=0$$

$$f_{ni}(\rho=L)=0$$

Discretization of the energy spectrum $\{E_i\}$

Each discrete state has wave function $\Psi_i^{(E_i)}$

$$\Psi_i^{(E_i)}=\frac{1}{\rho^{5/2}}\sum_n f_{ni}(\rho)\Phi_n(\rho,\Omega)$$

$$\Delta E=\frac{2\pi}{L}\frac{E_i}{\kappa_i}$$

$$\kappa_i=\sqrt{2mE_i}/\hbar$$

- ✓ For each *discrete* energy we get **only one wave function** associated to it.
- ✓ What incoming channel does it correspond to?
- ✓ Where are the other components corresponding to that energy?
- ✓ **Are we missing part of the total w.f.?**

$$\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\\ \vdots\\ \Psi_N\end{pmatrix}=\frac{1}{\rho^{5/2}}\begin{pmatrix}f_{11} & f_{21} & \dots & f_{N1}\\ f_{12} & f_{22} & \dots & f_{N2}\\ \vdots & \vdots & \vdots & \vdots\\ f_{1N} & f_{2N} & \dots & f_{NN}\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_N\end{pmatrix}=\frac{1}{\rho^{5/2}}R\Phi$$

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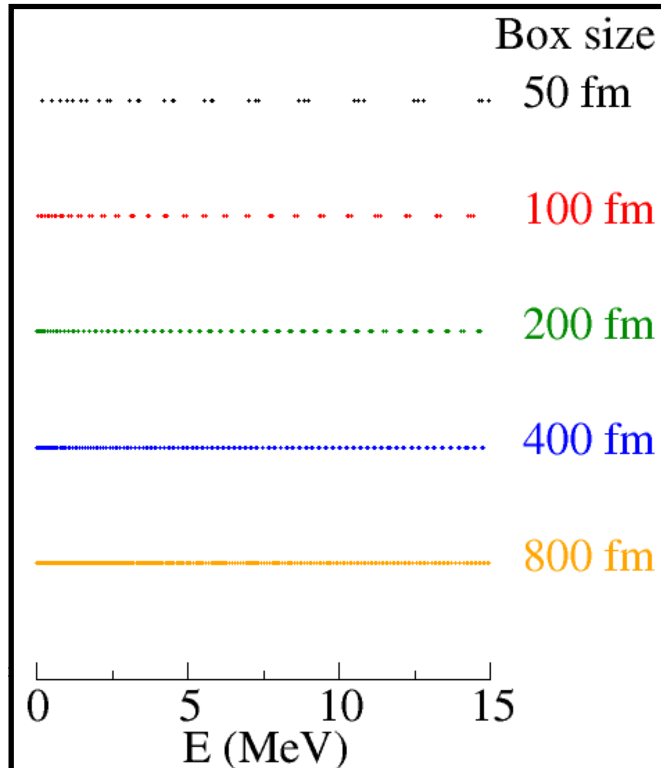
- ✓ The discrete states appear in bins of states.
- ✓ Each bin contains as many states as open channels in the calculation.
- ✓ The bigger the box the closer to degeneracy the states in the bin.
- ✓ The energy separation between the bins follows the *ΔE* rule.

$$\Delta E = \frac{2\pi}{L}\frac{E_i}{\kappa_i}$$

$$\kappa_i = \sqrt{2mE_i}/\hbar$$

**For *L* = 400 fm:**

- ✓ $E_i$ = {79.17,79.23,79.27} MeV
- ✓ $E_{i+1}$ = {79.81,79.87,79.89} MeV
- ✓ For $E_i$=80 MeV → $\Delta E$=0.6 MeV

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**Therefore:**

✓ The discrete states belonging to each bin of energies do actually describe the continuum state for all the possible incoming channels for the energy of the bin.

$$\Delta E=\frac{2\pi}{L}\frac{E_i}{\kappa_i}$$

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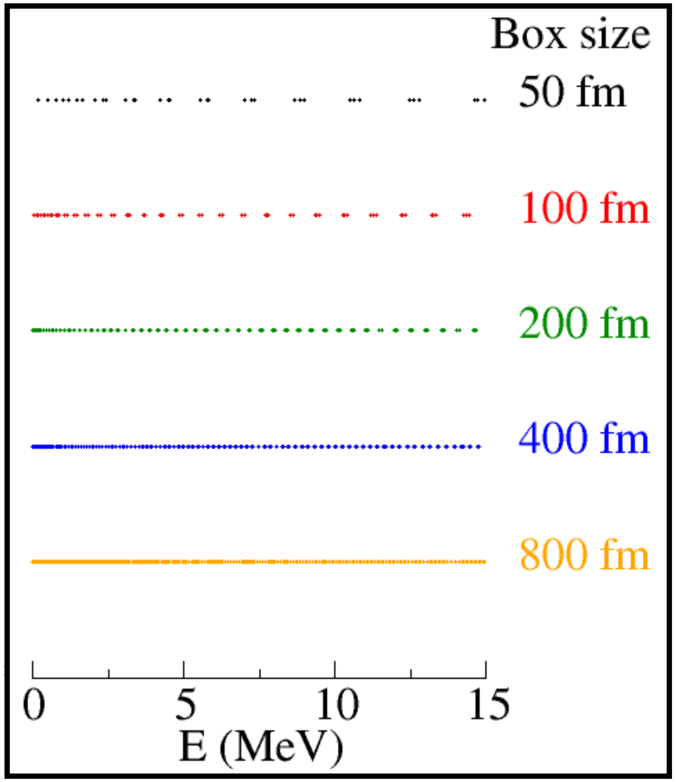


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Box boundary condition

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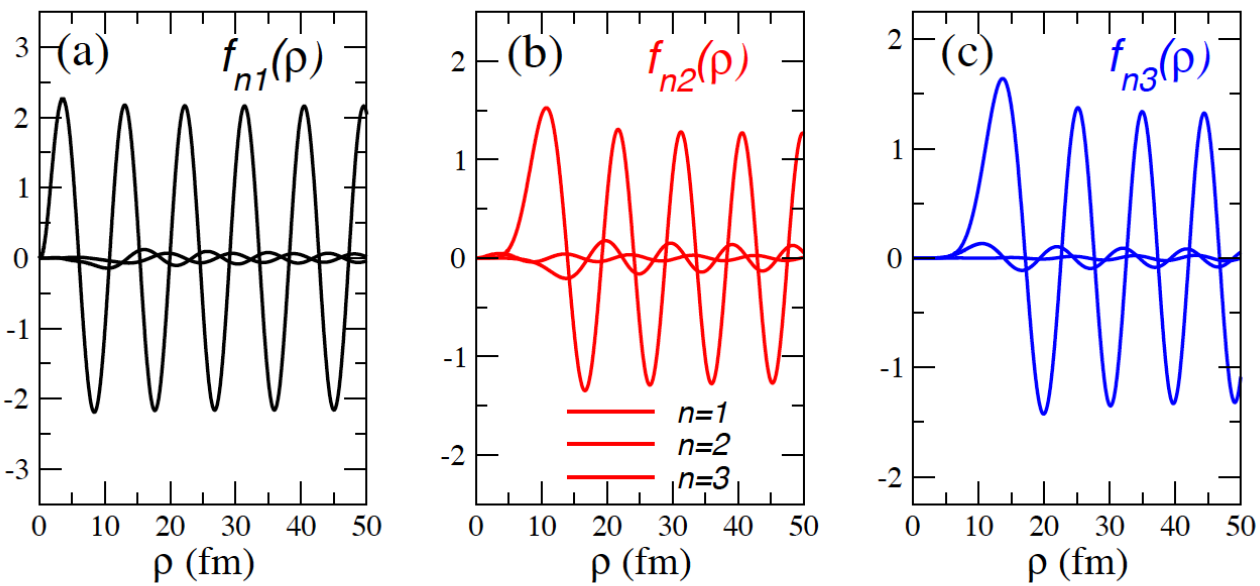


Radial matrix for the $0^+$ state of three spinless identical bosons ($Z$=0)

$E$=10 MeV

***No discretization***

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11} & f_{21} & \dots & f_{N1} \\ f_{12} & f_{22} & \dots & f_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N} & f_{2N} & \dots & f_{NN} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R\Phi$$

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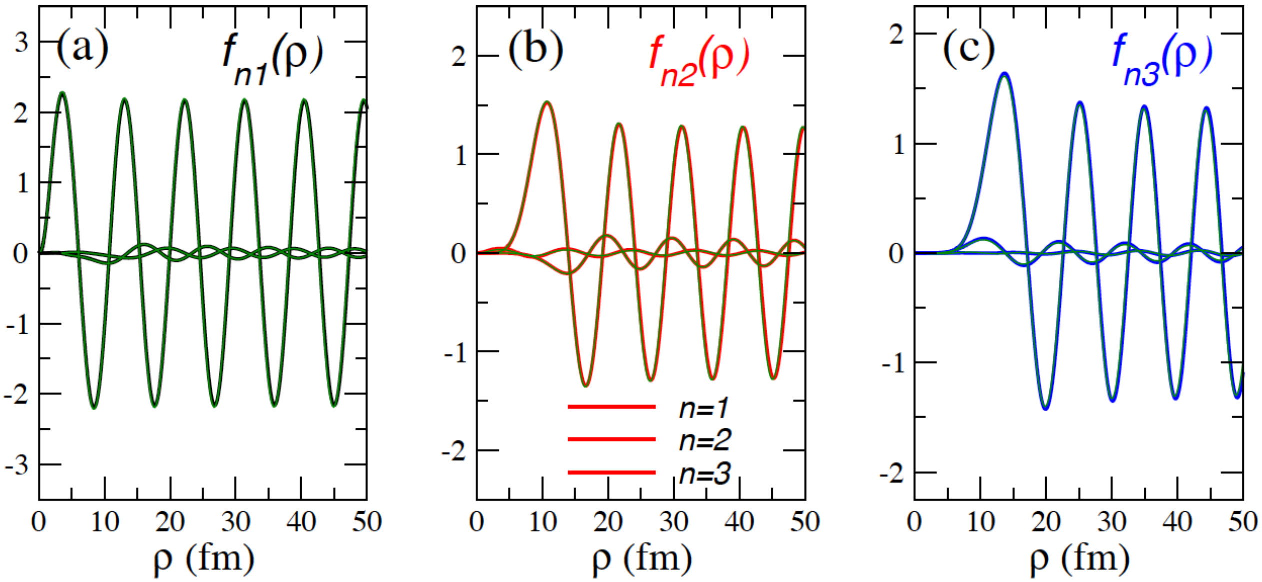


Radial matrix for the $0^+$ state of three spinless identical bosons ($Z$=0)

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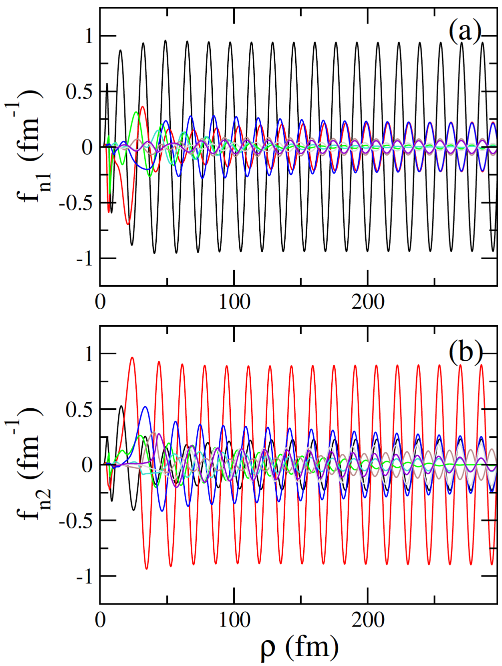
✓ Both methods are **fully equivalent**

✓ When discretizing the spectrum **no information is lost**

✓ The discretization method can be safely used for **charged** particles

# The three-alpha system




$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11} & f_{21} & \dots & f_{N1} \\ f_{12} & f_{22} & \dots & f_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N} & f_{2N} & \dots & f_{NN} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R\Phi$$

- ✓ First two rows of the radial matrix for the three-alpha system
- ✓ 8 adiabatic terms
- ✓ *E*=3.5 MeV
- ✓ Ali-Bodmer alpha-alpha potential

$$\Psi \to AF + BG$$
$$A = -\frac{2m}{\hbar^2}\langle \Psi | \hat{\mathcal{H}} - E | G \rangle$$
$$B = \frac{2m}{\hbar^2}\langle \Psi | \hat{\mathcal{H}} - E | F \rangle$$
$$K = -A^{-1}B$$
$$S = (1 + iK)(1 - iK)^{-1}$$

- ✓ *F* and *G* are the free solutions of the Hamiltonian
- ✓ The matrices *A* and *B* can be obtained by using the integral relations: E.G., C. Romero-Redondo, A. Kievsky, M. Viviani , PRA 86 (2012) 052709
- ✓ Determination of the *S*-matrix requires knowledge of the Coulomb matrices *F* and *G*

## The three-alpha system

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$$F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f^{reg}_{11} & f^{reg}_{21} & \dots & f^{reg}_{N1} \\ f^{reg}_{12} & f^{reg}_{22} & \dots & f^{reg}_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ f^{reg}_{1N} & f^{reg}_{2N} & \dots & f^{reg}_{NN} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{reg}\Phi$$

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f^{irr}_{11} & f^{irr}_{21} & \dots & f^{irr}_{N1} \\ f^{irr}_{12} & f^{irr}_{22} & \dots & f^{irr}_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ f^{irr}_{1N} & f^{irr}_{2N} & \dots & f^{irr}_{NN} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{irr}\Phi$$

- ✓ For **short-range** interactions $R_{reg}$ and $R_{irr}$ are diagonal, and they are given by the regular and irregular Bessel functions.
- ✓ In the **Coulomb** case $R_{reg}$ and $R_{irr}$ have to be computed numerically

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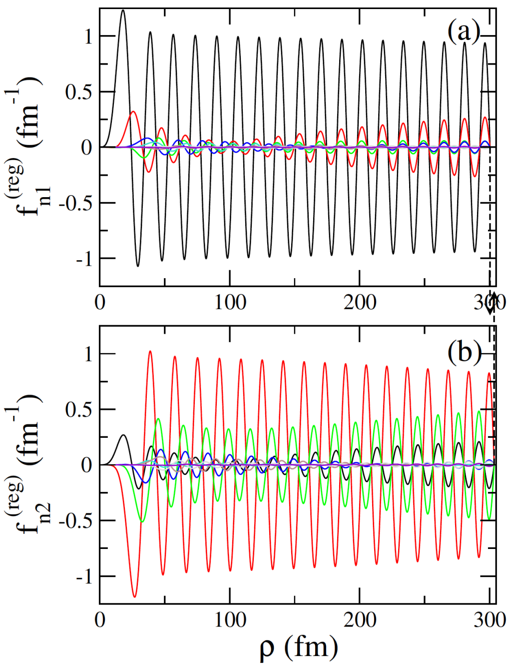


$$F=\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{reg} & f_{21}^{reg} & \dots & f_{N1}^{reg} \\ f_{12}^{reg} & f_{22}^{reg} & \dots & f_{N2}^{reg} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{reg} & f_{2N}^{reg} & \dots & f_{NN}^{reg} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{reg}\Phi$$

- ✓ The *F* matrix is computed exactly as the full solution but including only the Coulomb interaction.

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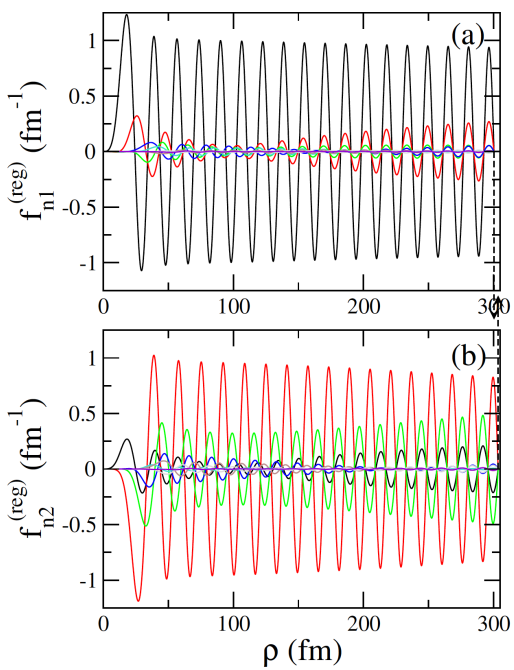


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✓ The *F* matrix is computed exactly as the full solution but including only the Coulomb interaction.

$$R_{reg} = U W_{reg} V^T$$

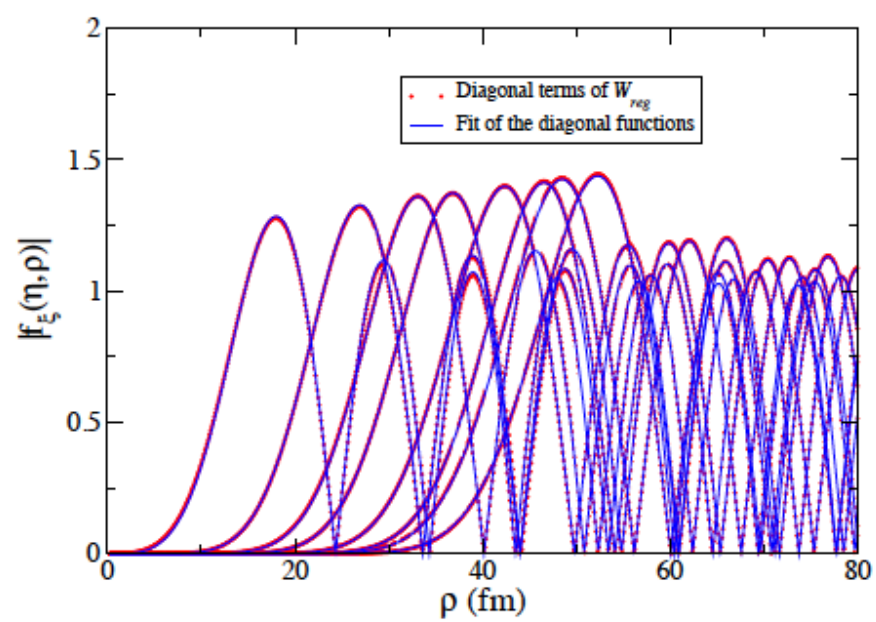
$W_{reg}$ is diagonal

The functions in the diagonal are regular Coulomb functions

$$W_{reg} \Rightarrow W_{irr}$$

$$R_{irr} = U W_{irr} V^T$$

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{irr} & f_{21}^{irr} & \dots & f_{N1}^{irr} \\ f_{12}^{irr} & f_{22}^{irr} & \dots & f_{N2}^{irr} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{irr} & f_{2N}^{irr} & \dots & f_{NN}^{irr} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{irr} \Phi$$



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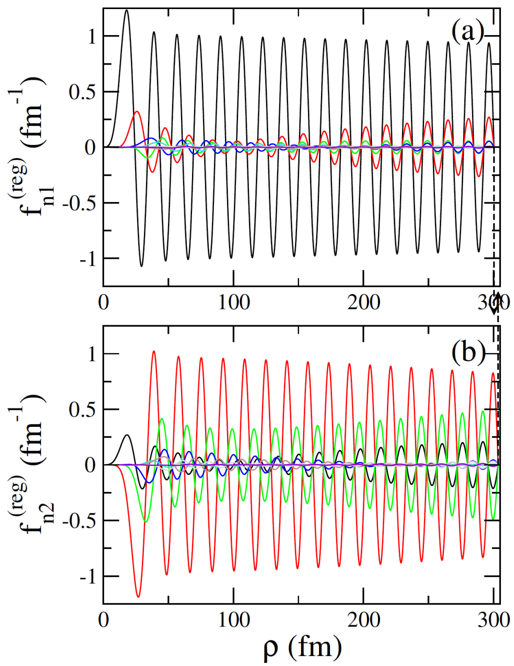


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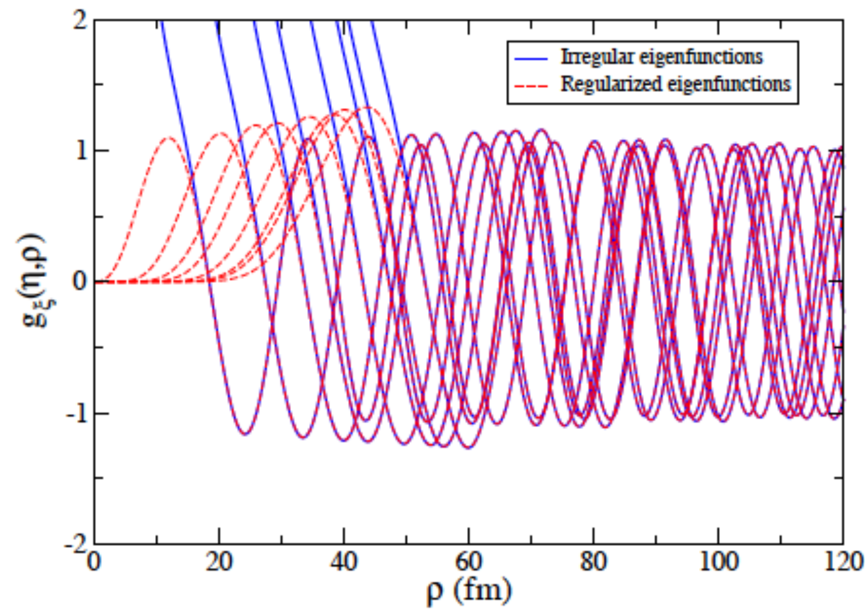
$$W_{reg} \Rightarrow W_{irr}$$

$$R_{irr} = U W_{irr} V^T$$

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{irr} & f_{21}^{irr} & \cdots & f_{N1}^{irr} \\ f_{12}^{irr} & f_{22}^{irr} & \cdots & f_{N2}^{irr} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{irr} & f_{2N}^{irr} & \cdots & f_{NN}^{irr} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{irr} \Phi$$



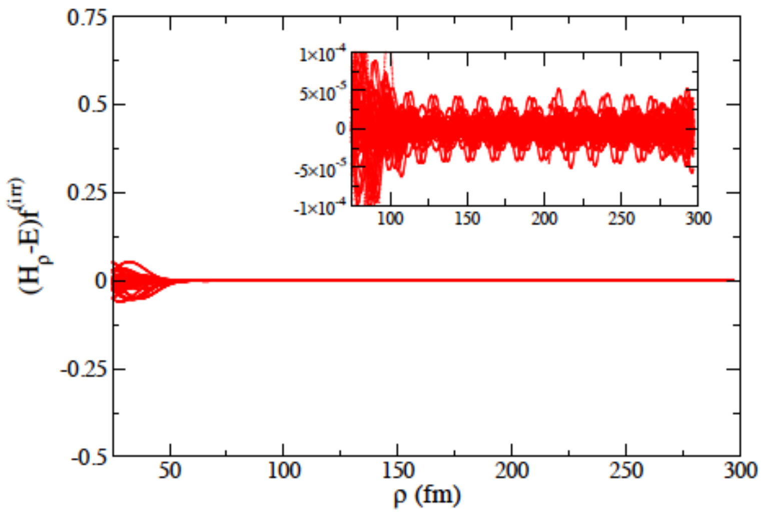
# The three-alpha system

$$F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{reg} & f_{21}^{reg} & \dots & f_{N1}^{reg} \\ f_{12}^{reg} & f_{22}^{reg} & \dots & f_{N2}^{reg} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{reg} & f_{2N}^{reg} & \dots & f_{NN}^{reg} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{reg} \Phi$$

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{irr} & f_{21}^{irr} & \dots & f_{N1}^{irr} \\ f_{12}^{irr} & f_{22}^{irr} & \dots & f_{N2}^{irr} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{irr} & f_{2N}^{irr} & \dots & f_{NN}^{irr} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{irr} \Phi$$

Test 1: Is the $R_{irr}$ matrix solution of the radial equations?

$$\left[-\frac{d^2}{d\rho^2} + \frac{2m}{\hbar^2}(V_n(\rho) - E)\right] f_{ni}(\rho) + \sum_{n'} \left(-2P_{nn'}\frac{d}{d\rho} - Q_{nn'}\right) f_{n'i}(\rho) = 0$$




$$\int d\rho \tilde{R}_{irr}^T R_{irr} = \mathbb{1}$$

**Fig. 7** (color online) Outer part: Result obtained when applying the set of differential radial equations (4) to the computed (regularized) irregular radial functions in $\tilde{R}_{irr}$. The inset shows the detail from $\rho = 50$ fm to $\rho = 300$ fm.

# The three-alpha system

$$F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{reg} & f_{21}^{reg} & \cdots & f_{N1}^{reg} \\ f_{12}^{reg} & f_{22}^{reg} & \cdots & f_{N2}^{reg} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{reg} & f_{2N}^{reg} & \cdots & f_{NN}^{reg} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{reg}\Phi$$

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{irr} & f_{21}^{irr} & \cdots & f_{N1}^{irr} \\ f_{12}^{irr} & f_{22}^{irr} & \cdots & f_{N2}^{irr} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{irr} & f_{2N}^{irr} & \cdots & f_{NN}^{irr} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{irr}\Phi$$

Test 2:

$$\left.\begin{aligned} F_L(\eta,\rho) &\to \sin\left(\rho - \eta\ln 2\rho - \frac{L\pi}{2}\right) \\ G_L(\eta,\rho) &\to \cos\left(\rho - \eta\ln 2\rho - \frac{L\pi}{2}\right) \end{aligned}\right\} \Longrightarrow R_{reg}^T R_{reg} + \tilde{R}_{irr}^T \tilde{R}_{irr} \to \mathbb{1}$$

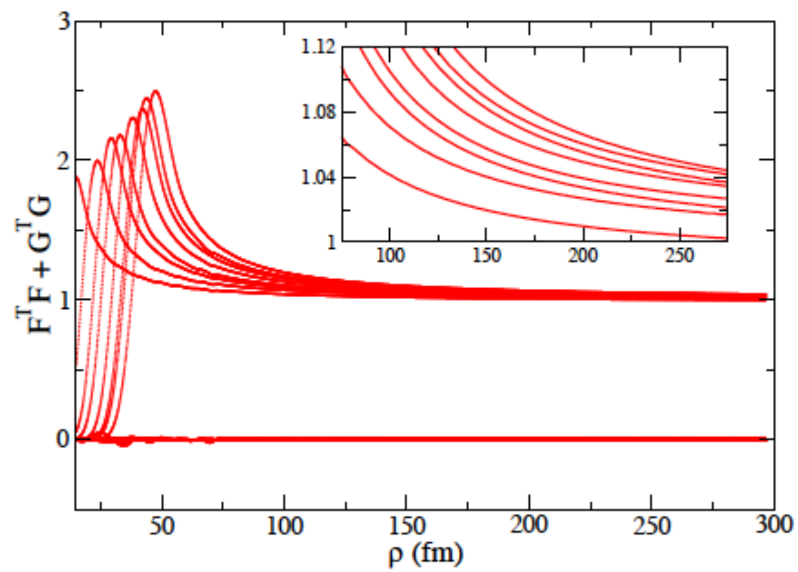



**Fig. 8** (color online) $R_{reg}^T R_{reg} + \tilde{R}_{irr}^T \tilde{R}_{irr}$ as a function of $\rho$. In the inner part a closer plot of the diagonal terms is shown.

# The three-alpha system

$$F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{reg} & f_{21}^{reg} & \dots & f_{N1}^{reg} \\ f_{12}^{reg} & f_{22}^{reg} & \dots & f_{N2}^{reg} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{reg} & f_{2N}^{reg} & \dots & f_{NN}^{reg} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{reg} \Phi$$

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{irr} & f_{21}^{irr} & \dots & f_{N1}^{irr} \\ f_{12}^{irr} & f_{22}^{irr} & \dots & f_{N2}^{irr} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{irr} & f_{2N}^{irr} & \dots & f_{NN}^{irr} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{irr} \Phi$$

**Test 3:** Wronskian relation

$$R'^{T}_{reg} \tilde{R}_{irr} - R^{T}_{reg} \tilde{R}'_{irr} \to \mathbb{1}$$

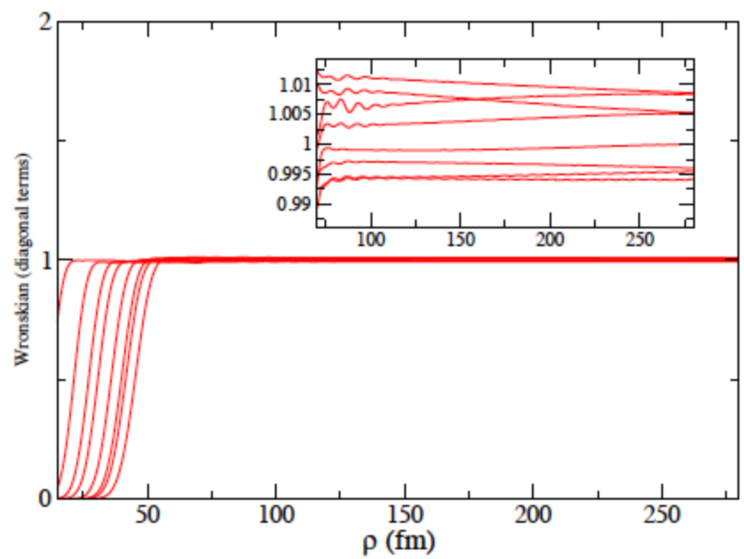



**Fig. 9** (color online) Diagonal terms of the Wronskian when using the regular and (regularized) irregular radial matrices $R_{reg}$ and $\tilde{R}_{irr}$. The inset shows the detail from $\rho = 70$ fm to $\rho = 300$ fm.

# The three-alpha system

$$F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{reg} & f_{21}^{reg} & \dots & f_{N1}^{reg} \\ f_{12}^{reg} & f_{22}^{reg} & \dots & f_{N2}^{reg} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{reg} & f_{2N}^{reg} & \dots & f_{NN}^{reg} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{reg} \Phi$$

$$G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{pmatrix} = \frac{1}{\rho^{5/2}} \begin{pmatrix} f_{11}^{irr} & f_{21}^{irr} & \dots & f_{N1}^{irr} \\ f_{12}^{irr} & f_{22}^{irr} & \dots & f_{N2}^{irr} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1N}^{irr} & f_{2N}^{irr} & \dots & f_{NN}^{irr} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_N \end{pmatrix} = \frac{1}{\rho^{5/2}} R_{irr} \Phi$$

Test 4: *S*-matrix

$$\left.\begin{aligned} A &= -\frac{2m}{\hbar^2}\langle \Psi | \hat{\mathcal{H}} - E | G \rangle \\ B &= \frac{2m}{\hbar^2}\langle \Psi | \hat{\mathcal{H}} - E | F \rangle \end{aligned}\right\} \Longrightarrow K = -A^{-1}B$$

$$S = (1 + iK)(1 - iK)^{-1} \Longrightarrow S^\dagger S = \mathbb{1}(\pm \sim 10^{-3})$$

$$R_{3\alpha}(E) = \frac{\hbar^5}{m_\alpha^3} \frac{144\sqrt{3}}{E^2} \sum_{J^\pi} \sum_{nn'} (2J+1) |S_{nn'}^{J^\pi}(E) - \delta_{nn'}|^2$$

## Summary:

- ✓ Three-body wave functions in the continuum require knowledge of the wave function for **all the possible incoming channels**.
- ✓ This can be done by imposing a **box boundary condition**, especially suitable when the asymptotic form of the wave function is not known analytically.
- ✓ This procedure can be used to obtain the **regular three-body Coulomb functions**.
- ✓ A method to obtain the **irregular partner** has been described.
- ✓ Different tests of the regular and irregular solutions have been shown.
- ✓ They can be used to extract the S-matrix of the process.