Spherically symmetric static solution

Schwarzschild metric

Schwarzschild metric describes the gravitational field outside a non-rotating spherical body like a star, planet, or black hole. It is a static, spherically symmetric solution of the vacuum Einstein equation ($R_{ab} = 0$).

The spherically symmetric static metric can be assumed to have the following form,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1}$$

where A and B are some yet unknown functions of radius r.

The following Maxima script,

```
derivabbrev:true; /* a better notation for derivatives */
load(ctensor); /* load the package to deal with tensors */
ct_coords:[t,r,o,p]; /* our coordiantes: t r theta phi */
depends([A,B],[r]); /* the functions A and B depend on r */
lg:ident(4); /* set up a 4x4 identity matrix */
lg[1,1]: A; /* g_{tt} = A */
lg[2,2]:-B; /* g_{rr} = -B */
lg[3,3]:-r^2; /* g_{\theta\theta} =  */
lg[4,4]:-r^2*sin(o)^2; /* g_{\phi\phi} */
cmetric(); /* compute the prerequisites for further calculations */
christof(mcs); /* calculates Christoffel symbols, mcs_{bca}=\Gamma^a_{bc} */
ricci(true); /* calculates the covariant symmetric Ricci tensor */
```

calculates analytically the Christoffel symbols[1] and the Ricci tensor[2] for this metric. Now, integrating[3] the vacuum Einstein equations, $R_{ab} = 0$, gives the famous Schwarzschild metric,

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2}$$

The integration constant r_g is determined from the Newtonian limit[4], $r_g = 2GM/c^2$, where M is the mass of the central body ($r_g = 2M$ in the units $G = c = 1$). It is called gravitational or Schwarzschild radius. The gravitational radius for the Earth is about 9mm, for the Sun – about 3km.

Motion in the Schwarzschild metric

Massive bodies Massive bodies move along geodesics, described by the geodesic equation

$$\frac{d}{ds}\left(g_{ab}u^b\right) = \frac{1}{2}g_{bc,a}u^b u^c\,. \tag{3}$$

[1] $\Gamma^r_{rr} = \frac{1}{2}\frac{B'}{B}$, $\Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A}$, $\Gamma^r_{tt} = \frac{1}{2}\frac{A'}{B}$, $\Gamma^\theta_{\theta r} = \frac{1}{r}$, $\Gamma^r_{\theta\theta} = -\frac{r}{B}$, $\Gamma^\phi_{\phi r} = \frac{1}{r}$, $\Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B}$, $\Gamma^\phi_{\phi\theta} = \cot\theta$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$.

[2]

$$R_{tt} = \frac{A''}{2B} + \frac{A'}{B}\left(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A}\right),$$

$$R_{\theta\theta} = 1 - \left(\frac{r}{B}\right)' - \frac{1}{2}\left(\frac{A'}{A} + \frac{B'}{B}\right)\frac{r}{B},$$

$$R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}.$$

[3] Making a linear combination $BR_{tt} + AR_{rr} = 0$ gives $A'B + AB' = 0 \Rightarrow AB = 1 \Rightarrow \frac{A'}{A} + \frac{B'}{B} = 0$. Then $R_{\theta\theta} = 0$ gives $B = \frac{1}{1-\frac{r_g}{r}}$, $A = 1 - \frac{r_g}{r}$, where r_g is an integration constant.

[4] $g_{00} \stackrel{r\to\infty}{\longrightarrow} 1 + 2\phi = 1 - 2\frac{GM}{r}$

For $a = t, \theta, \phi$ the corresponding equations in the Schwarzschild metric (2) are

$$\frac{d}{ds}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{ds}\right] = 0\ , \tag{4}$$

$$\frac{d}{ds}\left[r^2\frac{d\theta}{ds}\right] = r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2\ , \tag{5}$$

$$\frac{d}{ds}\left[r^2\sin^2\theta\frac{d\phi}{ds}\right] = 0\ . \tag{6}$$

The r-equation can be conveniently obtained by dividing the Schwarzschild metric (2) by ds^2,

$$1 = \left(1-\frac{2M}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right]\ . \tag{7}$$

Considering equatorial motion, the first three equations can be integrated as

$$\theta = \frac{\pi}{2}\ ,\quad r^2\frac{d\phi}{ds} = J\ ,\quad \left(1-\frac{2M}{r}\right)\frac{dt}{ds} = E\ , \tag{8}$$

where J and E are constants. The fourth equation then becomes

$$1 = \frac{E^2}{1-\frac{2M}{r}} - \frac{\frac{J^2}{r^4}r'^2}{1-\frac{2M}{r}} - \frac{J^2}{r^2}\ , \tag{9}$$

where $r' \equiv \frac{dr}{d\phi}$. Traditionally one makes a variable substitution $r = 1/u$,

$$(1-2Mu) = E^2 - J^2u'^2 - J^2u^2(1-2Mu), \tag{10}$$

and then differentiates the equation once. This gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2\ . \tag{11}$$

In this form the last term is a relativistic correction to the otherwise non-relativistic equation.

Light rays. The rays of light travel along the null-geodesics where $ds^2 = 0$. Consequently instead of ds one needs to use some parameter $d\lambda$ in the geodesic equations $\frac{Dk^a}{d\lambda} = 0$, where $k^a = \frac{dx^a}{d\lambda}$ and also the unity in the left-hand side of equation (7) has to be substituted with zero. This immediately leads to the equation

$$u'' + u = 3Mu^2\ , \tag{12}$$

which describes the trajectory of a ray of light in the Schwarzschild metric.

In the absence of the central body, $M = 0$ the space becomes flat, and equation (12) turns into equation for a straight line.

Exercises

1. Consider a non-relativistic equatorial ($\theta = \pi/2$) motion of a planet with mass m around a star with mass M described by a Lagrangian[5]

   $$\mathcal{L} = \frac{1}{2}m(\dot r^2 + r^2\dot\phi^2) + \frac{mM}{r}\ ,$$

   Write down the Euler-Lagrange equations,

   $$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot q} = \frac{\partial\mathcal{L}}{\partial q}\ ,$$

   for $q = r$ and ϕ. Using the first integral $r^2\dot\phi = J$ rewrite the r-equation as an equation for the function $u(\phi)$, where $u = 1/r$, and compare with (11).

[5] where the dot denotes the temporal derivative, $\dot r \equiv \frac{dr}{dt}$.

2. Show that in Newtonian mechanics an equatorial ($\theta = \pi/2$) trajectory of a light ray is described by the equation
$$u'' + u = 0\,,$$
where $u \doteq \frac{1}{r}$ and $u' \doteq \frac{du}{d\phi}$.

3. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. Answer: $r = \frac{3}{2}(2M)$.

4. Show, that in the Newtonian limit, $g_{00} = 1 + 2\phi/c^2$, the geodesic equation,
$$\frac{du^a}{ds} = -\Gamma^a_{bc} u^b u^c\,,$$
is consistent with the Newton's equation of motion,
$$\ddot{\vec{r}} = -\vec{\nabla}\phi\,.$$