Newtonian limit

In this section we shall show that in the limit of weak gravitational fields and slow motion General Relativity reduces to Newtonian gravitation.

Newtonian gravitation

The Newton's law of gravitation states that two particles with masses m and M located at a relative distance r attract each other with the force

$$F = G\frac{mM}{r^2}, \tag{1}$$

where $G \approx 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$ is the gravitational constant (first measured by Cavendish).

The force is *conservative* and allows potential formulation: the body M creates a gravitational potential ϕ,

$$\phi(\vec{r}) = -\frac{GM}{r}\,, \tag{2}$$

in which the particle m acquires a potential energy $m\phi$ with the corresponding force

$$\vec{F} = -m\nabla\phi\,. \tag{3}$$

The potential (2) satisfies the Poisson equation[1]

$$\nabla^2\phi(\vec{r}) = 4\pi GM\delta(\vec{r})\,. \tag{4}$$

If instead of a single point-mass with mass density $M\delta(\vec{r})$ there is a distribution of masses with density $\mu(\vec{r})$, the gravitational potential created by these masses satisfies the Poisson equation

$$\nabla^2\phi = 4\pi G\mu\,. \tag{5}$$

Equation (3) can be cast into a variational form with the action

$$\begin{aligned} S &= \int dt\left(\frac{1}{2}mv^2 - m\phi - mc^2\right) \\ &= -mc\int dt\left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right). \end{aligned} \tag{6}$$

Comparing with $S = -mc\int ds$ we get (squaring and dropping terms negligible in the limit $c \to \infty$)

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2dt^2 - d\vec{r}^2\,. \tag{7}$$

Thus in the Newtonian limit the metric tensor can be approximated[2] by $g_{ab} = \eta_{ab} + h_{ab}$, where η_{ab} is the Minkowski metric tensor and h_{ab} is a small correction, and the g_{00} component is given as

$$g_{00} = 1 + \frac{2\phi}{c^2}\,, \tag{8}$$

where ϕ satisfies the Poisson equation (5).

[1] $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})\,.$

[2] where we have neglected the terms $g_{\alpha\beta},\ \alpha\beta = 1,2,3$ since their contribution to ds^2 is not multiplied by c^2 and is thus negligible compared to the contribution from g_{00}.

Newtonian limit of general relativity

For a distribution of (otherwise non-interacting) masses with mass density $\mu(\vec{r})$ the energy-momentum tensor is

$$T_{ab} = \mu u_a u_b \,. \tag{9}$$

In the Newtonian limit, where all fields are weak and all velocities are small, $u_a = \{1, 0, 0, 0\}$, only the $_{00}$ component of the energy-momentum tensor is non-vanishing,

$$T_{00} = \mu \tag{10}$$

Therefore we shall only consider the $_{00}$ component of the Einstein's equation,

$$R_{00} = \kappa(T_{00} - \frac{1}{2}g_{00}T) = \frac{1}{2}\kappa\mu \,. \tag{11}$$

In the slow-weak limit all second order terms and temporal derivatives must be neglected altogether. The $_{00}$ component of the Ricci tensor then reduces to

$$R_{00} \doteq R^a_{0a0} = \Gamma^\alpha_{00,\alpha} \tag{12}$$

where the Greek symbols run over $1, 2, 3$.

Assuming $g_{00} = 1 + 2\phi$ and dropping the temporal derivatives, the Christoffel symbol becomes

$$\Gamma^\alpha_{00} = -\phi^{,\alpha} \,, \tag{13}$$

The Ricci tensor in the same limit is given as

$$R_{00} = -\phi^{;\alpha}_{,\alpha} \equiv \nabla^2\phi \,, \tag{14}$$

The Einstein equation thus turns into the Poisson's equation

$$\nabla^2\phi = \frac{1}{2}\kappa\mu \tag{15}$$

which is equivalent to the Newtonian theory if we put

$$\kappa = \frac{8\pi G}{c^4} \,. \tag{16}$$

Gravitational waves

In a weak gravitational field the space-time is almost flat and the metric tensor g_{ab} is equal to the flat metric η_{ab} plus a small correction h_{ab},

$$g_{ab} \;=\; \eta_{ab} + h_{ab} \,. \tag{17}$$

The Riemann tensor to the lowest order in h_{ab} is

$$R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}). \tag{18}$$

If we choose coordinates such that

$$(h^a_b - \frac{1}{2}h\delta^a_b)^{,b} = 0, \tag{19}$$

the Ricci tensor is simply

$$R_{ab} = -\frac{1}{2}h^{,c}_{ab,c} \tag{20}$$

and the vacuum Einstein's equation, $R_{ab} = 0$, turns into the ordinary wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) h_{ab} = 0 \,.$$

The intensity of gravitational radiation by a system of slowly moving bodies is determined by its quadrupole moment $D_{\alpha\beta}$

$$-\frac{dE}{dt} = \frac{G}{45c^5}(D^{'''}_{\alpha\beta})^2 \,. \tag{21}$$

Exercises

1. Consider dust (non-interacting incoherent matter) – a good approximation for our universe just at the moment. Argue, that its energy-momentum tensor is given as

$$T^{ab} = \mu u^a u^b \,,$$

where μ is the mass density of the dust measured by a co-moving observer, and $u^a = dx^a/ds$. Use the following strategy:

   (a) Argue, that it is a generally covariant tensor.

   (b) Argue that in special relativity T^{00} is the energy density.

   (c) Argue that in special relativity this tensor satisfies the equations,

$$T^{ab}{}_{,b} = 0 \,,$$

   by arguing that the 0-component, $T^{0b}{}_{,b} = 0$, represents the energy conservation law,

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot (\epsilon \vec{v}) = 0 \,,$$

   and that the spatial components, $T^{\alpha b}{}_{,b} = 0$, where $\alpha = 1, 2, 3$ represent the Navier-Stokes equation for the dust motion,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = 0 \,.$$

2. Show that from the metric

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - d\vec{r}^2$$

it follows, that time runs differently at different places in a gravitational potential and estimate the difference in the clock rates at the sea level and on top of the Everest mountain.