Action in general relativity

In general relativity the gravitational field is the metric tensor g_{ab} as indeed the derivatives of this tensor enter the equations of motion of bodies as apparent gravitational forces.

The equations that determine the gravitational field shall be obtained from the variational principle with the action in the form

$$S = S_M + S_G\,, \tag{1}$$

where S_G is the action for the gravitational field, to be defined later, and S_M is the action for matter[1].

Unlike electromagnetism, where there is an explicit coupling term between electromagnetic fields and charges, in general relativity there is no explicit coupling terms between gravitation and matter.

Indeed, in general relativity gravitation is the geometry of space-time rather than a matter field like the electromagnetic field. The action of the matter has the same form as in special relativity, although written in a generally covariant form.

The matter then couples to gravitation through the metric tensors in the matter action.

The variational principle to determine the gravitational field, created by a given distribution of matter, states that the actual gravitational field, g_{ab}, is that where a small variation of the field, $g_{ab} + \delta g_{ab}$ leads to vanishing variation of the action,

$$\delta S\big|_{g_{ab} \to g_{ab} + \delta g_{ab}} = 0\,. \tag{2}$$

Action for matter

The action for matter in general relativity has the same form as in special relativity, only rewritten, if needed, in a generally covariant way. Particularly, one has to substitute

$$d\Omega \quad \to \quad \sqrt{-g}d\Omega\,, \tag{3}$$
$$\partial^a \varphi \quad \to \quad g^{ab}\partial_b \varphi\,, \tag{4}$$
$$\partial_a A^b \quad \to \quad D_a A^b\,. \tag{5}$$

For example, the action for the electromagnetic field becomes

$$-\frac{1}{16\pi}\int F^{ab}F_{ab}\sqrt{-g}d\Omega\,, \tag{6}$$

the coupling between the electromagnetic field and currents,

$$-\int A_a j^a \sqrt{-g}d\Omega\,, \tag{7}$$

the action for a hypothetical scalar field ϕ,

$$-\int g^{ab}\partial_a \varphi \partial_b \varphi \sqrt{-g}d\Omega\,. \tag{8}$$

Energy-momentum tensor of matter

The action for matter is typically written in terms of the Lagrangian of the matter, $\mathcal{L}$,

$$S_M = \int \mathcal{L}\sqrt{-g}d\Omega\,. \tag{9}$$

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[1] Matter in general relativity is everything else but the gravitational field.

The variation of this action under a variation $g^{ab} \to g^{ab} + \delta g^{ab}$ of the metric tensor can be written in terms of a symmetric tensor T_{ab},

$$\delta S_M \doteq \frac{1}{2} \int T_{ab} \delta g^{ab} \sqrt{-g} d\Omega = -\frac{1}{2} \int T^{ab} \delta g_{ab} \sqrt{-g} d\Omega,$$

where[2]

$$\frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab} \doteq \delta(\sqrt{-g}\mathcal{L}). \tag{10}$$

The tensor T_{ab} is actually the energy-momentum tensor, since in a flat space it satisfies a conservation law. Indeed, consider an infinitesimal coordinate transformation,

$$x^a \to x^{'a} = x^a + \epsilon^a. \tag{11}$$

The variation of the metric tensor under this transformation can be written as[3]

$$\delta g^{ab} = \epsilon^{a;b} + \epsilon^{b;a}, \ \delta g_{ab} = -\epsilon_{a;b} - \epsilon_{b;a}. \tag{16}$$

The variation of the action then takes the form

$$\delta S = \int T_{ab}\epsilon^{a;b}\sqrt{-g}d\Omega. \tag{17}$$

Integrating by parts[4],

$$\delta S = -\int T^b_{a;b}\epsilon^a\sqrt{-g}d\Omega. \tag{18}$$

The variation of the action under coordinate transformation is zero. Thus the tensor T^a_b satisfies the equation

$$T^{ab}_{;b} = 0, \tag{19}$$

which in a flat space turns into the energy-momentum conservation equation $T^{ab}_{,b} = 0$. One can thus assume that the tensor is proportional to the canonical energy-momentum tensor. Direct calculations show that the proportionality factor is equal unity.

Exercises

1. The Lagrangian density for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{16\pi}F_{ab}F^{ab}.$$

   Calculate the corresponding energy-momentum tensor using the usual definition,

$$\delta\left(\sqrt{-g}\mathcal{L}\right)_{g^{ab}\to g^{ab}+\delta g^{ab}} = \frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab}.$$

   Answer:

$$T_{ab} = \frac{1}{16\pi}\left(F_{cd}F^{cd}g_{ab} - 4F_{ac}F_b{}^c\right).$$

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[2]From $g_{ab}g^{bc} = \delta_a^c$ follows $g_{ab}\delta g^{bc} = -\delta g_{ab}g^{bc}$ and therefore $T_{ab}\delta g^{ab} = -T^{ab}\delta g_{ab}$.

[3]Differentiating $x'^a = x^a + \epsilon^a$ gives

$$x'^a_{,b} = \delta^a_b + \epsilon^a_{,b}. \tag{12}$$

Substituting this into the transformation rule for the metric tensor,

$$g_{ab}(x) = x'^c_{,a}x'^d_{,b}g_{cd}(x'), \tag{13}$$

gives

$$g_{ab}(x) = (\delta^c_a + \epsilon^c_{,a})(\delta^d_b + \epsilon^d_{,b})\left(g'_{cd}(x) + g_{cd,e}\epsilon^e + O((\epsilon)^2)\right) \tag{14}$$

$$= g'_{ab} + g_{ad}\epsilon^d_{,b} + g_{bd}\epsilon^d_{,a} + g_{ab,e}\epsilon^e + O\left((\epsilon)^2\right) = g'_{ab} + \epsilon_{a;b} + \epsilon_{b;a} + O\left((\epsilon)^2\right). \tag{15}$$

[4]using

$$\sqrt{-g}A^c_{;c} = (\sqrt{-g}A^c)_{,c}$$

the total covariant didifferential does not contribute to variation.

2. In the Minkowski space consider a scalar field φ with action

$$S = \int d\Omega \left(\frac{1}{2} \varphi^{,a} \varphi_{,a} - \frac{1}{2} m^2 \varphi^2 \right)$$

and calculate its "translation-invariance" energy-momentum tensor,

$$T_b^a = \frac{\partial \mathcal{L}}{\partial \varphi_{,a}} \varphi_{,b} - \mathcal{L} \delta_b^a \ .$$

Rewrite the action in a generally covariant form and calculate its "metric" energy-momentum tensor,

$$\delta \left(\sqrt{-g} \mathcal{L} \right)_{g^{ab} \to g^{ab} + \delta g^{ab}} = \frac{1}{2} \sqrt{-g} T_{ab} \delta g^{ab}$$

Discuss the results.

3. Show that for infinitesimally small coordinate transformation

$$x^a \to x^a + \epsilon^a$$

the variation of the metric tensor is given as

$$\delta g^{ab} = \epsilon^{a;b} + \epsilon^{b;a} \ , \ \delta g_{ab} = -\epsilon_{a;b} - \epsilon_{b;a} \ .$$

4. Derive the energy-momentum tensor of a particle with mass m.

5. Argue that in the flat space the equation $j^a_{,a}$ represents a conservation law.