The principle of stationary action

Action is an attribute of the dynamics of a physical system. It is a functional which takes the trajectory of the system (also called *path* or *history*) as its argument and returns a covariant real scalar as the result[1]. Generally, the action takes different values for different paths.

Classical mechanics postulates that

> *The path actually followed by a physical system is that for which the action is stationary, that is, its variation vanishes,*
>
> $$\delta S = 0 \,. \tag{2}$$

Vanishing variation means the action has an extremum on the actual trajectory. The postulate is called the *principle of stationary action* or the *variational principle*.

The classical equations of motion can be derived from the variational principle.

Motion of a charged body in gravitational and electromagnetic fields

In special relativity the action of a body with mass m and charge e, moving in a given electromagnetic field A^a, is given (in the units $c = 1$) as

$$S = -m \int ds - e \int dx^a A_a \,, \tag{3}$$

where the integrals are taken along the trajectory of the body. In this form the action is generally covariant and can be directly used in general relativity. One only has to remember that the metric tensor is not constant throughout the curved space.

We have to calculate the variation of this action under a small variation of the trajectory of the body,

$$x^a \to x^a + \delta x^a \,. \tag{4}$$

The variation of the first term in (3) has already been calculated in the section about geodesics,

$$\delta \left(-m \int ds \right) = \int ds \delta x^a m \left(\frac{du_a}{ds} - \frac{1}{2} g_{bc,a} u^b u^c \right) \,. \tag{5}$$

The variation of the term $dx^a A_a$ is given as

$$\delta(dx^a A_a) = \delta dx^a A_a + u^a ds A_{a,b} \delta x^b = \delta dx^a A_a + u^b ds A_{b,a} \delta x^a \,. \tag{6}$$

As usual the term $\delta dx^a A_a$ is integrated by parts,

$$\delta dx^a A_a = d(\delta x^a A_a) - \delta x^a dA_a = d(\delta x^a A_a) - \delta x^a A_{a,b} u^b ds \,. \tag{7}$$

Inserting this into (6) leads to

$$\delta(dx^a A_a) = d(\delta x^a A_a) + (-A_{a,b} + A_{b,a}) u^b ds \delta x^a \,. \tag{8}$$

The expression in parentheses is called the *electromagnetic tensor*,

$$F_{ab} \doteq -A_{a,b} + A_{b,a} \,. \tag{9}$$

[1] If the action is represented as an integral over time, taken along the path of the system between the initial time and the final time of the development of the system,

$$S = \int \mathcal{L} dt \,, \tag{1}$$

the integrand $\mathcal{L}$ is called the *Lagrangian*.

It is apparently antisymmetric, $F_{ab} = -F_{ba}$. Although it contains non-covariant derivatives it is actually a tensor since in a torsion free space it can be also written through covariant derivatives,

$$F_{ab} = -A_{a;b} + A_{b;a} \,. \tag{10}$$

The full differentials in (8) as usual does not contribute to the variation since δx^a is zero at the end-points of the trajectory. The variation of the action (3 then becomes

$$\delta S = \int ds \delta x^a \left(m\frac{du_a}{ds} - m\frac{1}{2}g_{bc,a}u^b u^c - eF_{ab}u^b \right) \,. \tag{11}$$

Since the variation δx^a is arbitrary, $\delta S = 0$ means the expression in parentheses has to vanish identically on physical trajectories, giving the the equation of motion of a charged body in both gravitational and electro-magnetic fields (the generalization of the *Lorentz force* equation),

$$m\frac{du_a}{ds} - m\frac{1}{2}g_{bc,a}u^b u^c = eF_{ab}u^b \,. \tag{12}$$

It can also be written as

$$m\frac{du_a}{ds} - m\Gamma_{bca}u^b u^c = eF_{ab}u^b \,, \tag{13}$$

or as

$$m\frac{Du_a}{ds} = eF_{ab}u^b \,. \tag{14}$$

Maxwell equations in curved spaces

In this section we shall derive the generally covariant equations for the electomagnetic field – the Maxwell equations in curved spaces.

The homogeneous Maxwell equation

The generally covariant form of the homogeneous Maxwell equation can be deduced from its form in Minkowski space,

$$F_{(ab,c)} = F_{ab,c} + F_{bc,a} + F_{ca,b} = 0 \,, \tag{15}$$

(parentheses denote cyclic permutaion of the included indices) where

$$F_{ab} \doteq -A_{a,b} + A_{b,a} \tag{16}$$

is the electromagnetic tensor. Actually, due to the symmetry of Christoffel symbols $\Gamma^a_{bc} = \Gamma^a_{cb}$ (and asymmetry of the electromagnetic tensor) both the electromagnetic tensor and the homogeneous Maxwell equation are already generally covariant and thus preserve their form in curved spaces since

$$-A_{\{a,b\}} = -A_{\{a;b\}} \,, \tag{17}$$

and

$$F_{(ab,c)} = F_{(ab;c)} \,. \tag{18}$$

The inhomogeneous Maxwell equation

For the inhomogeneous Maxwell equation we shall first deduce the generally covariant action of the electromagnetic field, and then derive the equation from the stationary action principle.

In Minkowski space the action of the electromagnetic field is given as an integral over the whole space-time,

$$S = \int d\Omega \left(-\frac{1}{16\pi}F^{ab}F_{ab} - A_a j^a \right) \,. \tag{19}$$

Covariant volume element The infinitesimal volume element,

$$d\Omega \doteq d^4x\,, \tag{20}$$

in this action is not invariant under a general coordinate transformation and has to be substituted with the generally covariant volume element,

$$d\Omega \to \sqrt{-g}d\Omega\,, \tag{21}$$

where g is the determinant of the metric tensor g_{ab} ($g < 0$). Indeed, the metric tensor transforms as

$$g_{ab} = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b}g'_{cd}\,. \tag{22}$$

Taking determinant of both sides gives $g = J'^2 g'$, or

$$\sqrt{-g} = J'\sqrt{-g'}\,, \tag{23}$$

where $J' = \left|\frac{\partial x'^a}{\partial x^b}\right|$ is the Jacobian determinant of the transformation. The 4-volume transforms as

$$d\Omega = \left|\frac{\partial x^a}{\partial x'^b}\right| d\Omega' = \frac{1}{J'}d\Omega'. \tag{24}$$

Apparently the combination $\sqrt{-g}d\Omega$ transforms as

$$\sqrt{-g}d\Omega = J'\sqrt{-g'}\frac{1}{J'}d\Omega' = \sqrt{-g'}d\Omega', \tag{25}$$

and is thus generally invariant.

The generally covariant action of the electromagnetic field is thus given as

$$S = \int \sqrt{-g}d\Omega\left(-\frac{1}{16\pi}F^{ab}F_{ab} - A_a j^a\right)\,. \tag{26}$$

Variation of the action and the inhomogeneous Maxwell equation The variation of the electromagnetic field, $A_a \to A_a + \delta A_a$, in the second term of the action produces

$$-\delta\int\sqrt{-g}d\Omega A_a j^a = -\int d\Omega\sqrt{-g}j^a\delta A_a\,. \tag{27}$$

The first term gives

$$-\frac{1}{16\pi}\delta\int\sqrt{-g}d\Omega F^{ab}F_{ab} = +\frac{1}{4\pi}\int d\Omega\sqrt{-g}F^{ab}(\delta A_a)_{,b} = -\frac{1}{4\pi}\int d\Omega\left(\sqrt{-g}F^{ab}\right)_{,b}\delta A_a \tag{28}$$

Combining the two terms gives the variation of the electromagnetic action,

$$\delta S = \int d\Omega\delta A_a\left(-\frac{1}{4\pi}\left(\sqrt{-g}F^{ab}\right)_{,b} - \sqrt{-g}j^a\right)\,, \tag{29}$$

from which directly follows the sought inhomogeneous Maxwell equation in a curved space,

$$\left(\sqrt{-g}F^{ab}\right)_{,a} = 4\pi\sqrt{-g}j^b\,. \tag{30}$$

It can also be written in an explicitely covariant form,

$$F^{ab}_{;a} = 4\pi j^b\,, \tag{31}$$

using the identity

$$\sqrt{-g}V^a_{;a} = (\sqrt{-g}V^a)_{,a}\,. \tag{32}$$

To prove that, we shall first need the formula for the variation δg of the determinat g of the metric tensor (the Jacobi's formula). Suppose we vary one element, say g_{23}, of the metric tensor. The contribution of this element to the determinant of the metric tensor, g, is given as

$$g_{23}C_{23} = g_{23}gg^{23}\,, \tag{33}$$

where C_{23} is the cofactor of the element g_{23}, and g^{23} is the element of the inverse metric tensor. Then, apparently[2],

$$\delta g = g\delta g_{ab}g^{ab} = -gg_{ab}\delta g^{ab}\,. \tag{34}$$

Now let us consider the covariant divergence of the electromagnetic tensor,

$$F^{ab}_{;a} = F^{ab}_{,a} + \Gamma^a_{da}F^{db}\,. \tag{35}$$

The contraction Γ^a_{da} of the Christoffel symbol is given as

$$\Gamma^a_{da} = \frac{1}{2}g^{ab}g_{ab,d} = \frac{1}{2g}g_{,d}\,. \tag{36}$$

The divergence then becomes

$$F^{ab}_{;a} = F^{ab}_{,a} + \frac{1}{2g}g_{,d}F^{db} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}F^{ab}\right)_{,a}\,, \tag{37}$$

which concludes the proof.

Exercises

1. Argue that in Minkowski space the electromagnetic field satisfies the homogeneous Maxwell equation,
$$F_{(ab,c)} \doteq F_{ab,c} + F_{bc,a} + F_{ca,b} = 0\,,$$
where
$$F_{ab} = -A_{a,b} + A_{b,c}\,.$$

2. Argue that in a curved space the electromagneric field satisfies the generally covariant form of this equation,
$$F_{(ab;c)} = 0\,,$$
where
$$F_{ab} = -A_{a;b} + A_{b;a} = -A_{a,b} + A_{b,a}\,.$$

3. Derive the second Maxwell equation in a curved space,
$$\left(\sqrt{-g}F^{ab}\right)_{,a} = 4\pi\sqrt{-g}j^b\,,$$
from the action
$$S = \int\left(-\frac{1}{16\pi}F^{ab}F_{ab} - A_aj^a\right)\sqrt{-g}d\Omega\,.$$
Show that the equation can also be written as
$$F^{ab}_{;a} = 4\pi j^b\,.$$
Hints:
   (a) show that $\Gamma^a_{ba} = \frac{1}{2g}g_{,b} = (\ln\sqrt{-g})_{,b}$
   (b) show that $F^{ab}_{;a} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ab})_{,a}$

[2] In matrix calculus This formula is called the Jacobi's formula

Show that from this equation it follows, that

$$\left(\sqrt{-g}j^a\right)_{,a} = 0 = j^a_{;a} \,.$$

4. Argue that the electromagnatic tensor is actually a tensor by proving that

$$F_{ab} \doteq -A_{a,b} + A_{b,a} = -A_{a;b} + A_{b;a} \,. \tag{38}$$

5. Argue that

$$\frac{du_a}{ds} - \frac{1}{2}g_{bc,a}u^b u^c = \frac{du_a}{ds} - \Gamma_{bca}u^b u^c \,. \tag{39}$$

6. Derive the Lorentz force equation from the action $S = -m\int ds - e\int dx^a A_a$ in the Minkowski space of special relativity. Rewrite this equation in 3-notation where

$$A^a = \{\phi, \vec{A}\}\,,\ \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\,,\ \vec{H} = \text{curl } \vec{A} \equiv \vec{\nabla} \times \vec{A}\,.$$

7. In Minkowski space of special relativity from the action

$$S = -\frac{1}{8\pi}\int d^4x A_{a,b}A^{a,b} - \int d^4x A^a j_a$$

derive[3] the Maxwell equations with sources,

$$A^{b,a}_{,a} = 4\pi j^b \,.$$

Show that with the Lorenz condition,

$$A^a_{,a} = 0\,,$$

it is equivalent to

$$F^{ab}_{,a} = 4\pi j^b \,.$$

Write down the latter in 3-notation.

[3]For the fields the usual “ingtegration by parts” is done using the Gauss theorem in Minkowski space,

$$\int_\Omega d^4x \frac{\partial B^a}{\partial x^a} = \oint_{\partial\Omega} B^a dS_a \,,$$

where dS_a is an infinitesimal element of the hyper-surface $\partial\Omega$ of the 4-volume Ω. In Eucledean 3D space it has a more familiar form,

$$\int_V dV \left(\vec{\nabla}\cdot\vec{B}\right) = \int_{\partial V} dS \left(\vec{B}\cdot\vec{n}\right).$$