Motion of free bodies in gravitational fields

Geodesic

In the general theory of relativity free bodies[1] move along special curves, called *geodesics*. A geodesic is a generalization of the notion of a *line* to curved spaces. Among all curves connecting two points geodesic is the curve with extremal measure. Geodesic is also the trajectory of a body moving with "constant velocity".

The term geodesic comes from *geodesy*, the science of measuring the size and shape of Earth. In the original sense, a geodesic was the shortest route between two points on the Earth's surface, namely, a segment of a great circle. The term has since been generalized to include measurements in more general mathematical spaces.

Geodesic as constant velocity trajectory

The velocity vector u^a of a moving body is defined as

$$u^a = \frac{dx^a}{ds}\,, \tag{1}$$

where dx^a is the infinitesimal movement of the body along the trajectory and $ds = \sqrt{g_{ab}dx^a dx^b}$ is the invariant interval.

In special relativity a free body moves with constant velocity such that the differential of the velocity along the trajectory vanishes,

$$du^a = 0\,. \tag{2}$$

This equation is not generally covariant and cannot by used in the curved space of general relativity. A suitable generalization to curved spaces is vanishing of the covariant differential of the velocity,

$$Du^a = 0\,. \tag{3}$$

This is actually the *geodesic equation*. It can be also written as

$$\frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0\,, \tag{4}$$

or

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0\,. \tag{5}$$

One can interpete this equation as (the relativistic generalization of) Newton's second law of motion: acceleration of the body equals the force acting on the body (divided by mass). Therefore the quantities $-\Gamma^a_{bc}u^b u^c$ appear as forces — inertial and gravitational — acting on the body. Since the Christoffel symbols (and hence the forces) are propotinal to derivatives of the metric tensor, one can say that the latter plays the role of the potential of gravitational forces. Thus the gravitational field is given by the metric tensor of the time-space and is thus a tensor field.

Geodesic as extremal trajectory

The measure μ of a trajectory (of a moving body) is defined as the sum of infinitesimal intervals ds along the trajectory,

$$\mu = \int ds\,. \tag{6}$$

The extremal trajectory is the one where the variation of the measure as function of the trajectory vanishes,

$$\delta\mu = 0\,. \tag{7}$$

[1] In general relativity a free body is a body free from all *physical forces*. Gravitational forces are equivalent to inertial forces and do not count as *physical*.

To calculate the variation of the measure we first vary the interval ds,

$$\delta ds = \delta\sqrt{g_{ab}dx^a dx^b} = \frac{1}{2}\frac{1}{\sqrt{g_{ab}dx^a dx^b}}\delta\left(g_{ab}dx^a dx^b\right) = \frac{1}{2}\frac{1}{ds}\left(\delta g_{ab}dx^a dx^b + 2g_{ab}\delta dx^a dx^b\right) . \quad (8)$$

Using $\delta g_{ab} = g_{ab,c}\delta x^c$ and $\frac{1}{ds}g_{ab}dx^b = u_a$ gives[2]

$$\delta ds = \frac{1}{2}g_{ab,c}u^a u^b \delta x^c ds + \delta dx^c u_c , \quad (9)$$

where in the second term the summation index a was replaced with c.

Assuming the functions are smooth enough we can exchange the order of differentials in the second term and integrate it by parts using

$$\delta dx^c u_c = d\delta x^a u_a = d(\delta x^a u_a) - \delta x^c du_c. \quad (10)$$

The full differential does not contribute to the variation, and we finally arrive at

$$\delta\mu = \int ds\delta x^c\left(-\frac{du_c}{ds} + \frac{1}{2}g_{ab,c}u^a u^b\right) = 0 . \quad (11)$$

Since the variation δx is arbitrary, it is the expression in parentheses that should be equal zero identically along the trajectory, which gives the following equation for the curve with extremal measure,

$$\frac{du_c}{ds} - \frac{1}{2}g_{ab,c}u^a u^b = 0. \quad (12)$$

This equation is equivalent to the no-acceleration equation (3)[3]. Thus the trajectory with extremal measure is also the trajectory along which the covariant differential of the velocity of freely moving body is zero. This is the consequence of the fact that the action S of a free body with mass m is propotional to the measure of its trajectory,

$$S = -mc\int ds . \quad (16)$$

Motion of a ray of light

The equation (3) is not applicable to the propagation of a ray of light since the inderval ds along the ray is always zero. In this case one has to use certain parameter, λ, which varies (smoothly) along the ray. Then one can introduce the wave-vector, $k^a = \frac{dx^a}{d\lambda}$. In special relativity the ray of light propagates along a line where $dk^a = 0$. In a curved space in analogy with (3) this becomes

$$Dk^a = 0 , \quad (17)$$

or

$$\frac{dk^a}{d\lambda} + \Gamma^a_{bc}k^b k^c = 0 . \quad (18)$$

These equations, together with the condition that for the ray of light always $k^a k_a \propto dx^a dx_a = ds^2 = 0$, also determine the parameter λ.

[2] As always,

$$f_{,a} \doteq \frac{\partial f}{\partial x^a} .$$

[3] Indeed, from

$$0 = Dg_{ab} = dg_{ab} - \Gamma^e{}_{ac}g_{eb}dx^c - \Gamma^e{}_{bc}g_{ae}dx^c = (g_{ab,c} - \Gamma_{bac} - \Gamma_{abc})dx^c \quad (13)$$

it follows that

$$g_{ab,c} = \Gamma_{bac} + \Gamma_{abc} . \quad (14)$$

Inserting this into (12) and using the symmetry $u^a u^b = u^b u^a$ gives

$$\frac{du_c}{ds} - \frac{1}{2}g_{ab,c}u^a u^b = \frac{du_c}{ds} - \frac{1}{2}\left(\Gamma_{bac}u^a u^b + \Gamma_{abc}u^a u^b\right) = \frac{du_c}{ds} - \Gamma_{abc}u^a u^b = \frac{Du_c}{ds} , \quad (15)$$

which had to be demonstrated.

Motion of free bodies in general relativity

In general relativity a free body — that is, a body affected only by inertial and gravitational forces — moves along a geodesic. Massive bodies do not create physical fields around them but rather distort space-time in their vicinity causing the geodesics to become curved.

Exercises

1. Argue that there is no difference between vectors with indices up and vectors with indices down in a Euclidean space with Cartesian coordinates.

2. Argue that the quantity $u^a u_a = g_{ab} u^a u^b$ is a (covariant) scalar and calculate its value.

3. Prove that (4) and (12) are equivalent.

4. Consider the parametric equations for a line in Cartesian coordinates x and y,

$$\frac{d^2x}{ds^2} = 0\,,\ \frac{d^2y}{ds^2} = 0\,. \tag{19}$$

   Make a coordinate transformation to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) and derive the corresponding equations in the r, θ coordinates. Prove that they are identical to the geodesic equation (5).

5. Consider a non-relativistic motion of a free body in a plane with Cartesian coordinates x and y. The equation of motion of the body is given as

$$\frac{dv_x}{dt} = 0\,,\ \frac{dv_y}{dt} = 0\,. \tag{20}$$

   Make a transformation to polar coordinates $\{r, \theta\}$ and derive the corresponding equations for the velocity components $\{v_r, v_\theta\}$.

   Prove that they are identical to the geodesic equations $Dv_r = 0$, $Dv_\theta = 0$.