Covariant differentiation

Vectors in curved spaces In everyday life—which spatially is largely restricted to a curved two-dimensional metric space: the surface of Earth—a vector is a small arrow pointing in a certain direction along the surface. For example, the arrow on a road-sign indicating the direction toward the nearest town. The arrow must obviously be small compared to (at least) the relief structures on the Earth's surface.

The important property of the vector is that, although it might point at different direction relative to your car, it always points in the direction of the town. In other words a vector has certain useful transformational properties when one changes ones reference frames.

Geometrically the arrow can be described by specifying the coordinates of the nock and the head the arrow. Actually, one only needs the coordinate differences between these two points, as the absolute position of the road-sign is not terribly important (as long as you can still see it from the car on the road).

The smallness of the arrow in mathematical language means that the coordinate difference is infinitesimal. In other words,

A vector—in the sence of a small pointing arrow—is a coordinate differential.

In pysics we use many different vectors, but they are all related to small arrows in the sence that under a change of the reference frame they all transform as coordinate differentials.

Coordinates and vectors A set of four numbers used to specify the location of an event in space-time[1] is called *coordinates* and is denoted as $\{x^0, x^1, x^2, x^3\}$, or as x^a, or simply as x.

Under a general coordinate transformation $x \leftarrow x'$ the coordinate differentials $dx \leftarrow dx'$ transform linearly,

$$dx^a = \sum_{b=0}^{3} \frac{\partial x^a}{\partial x'^b} dx'^b \ . \tag{1}$$

through the Jacobian matrix $J \equiv \frac{\partial x'^b}{\partial x^a}$.

There exist another object that transforms linearly – the set of partial derivatives of a scalar function $\phi(x)$. It transforms through the inverse Jacobian matrix[2] $J^{-1} = \frac{\partial x^a}{\partial x'^b}$,

$$\frac{\partial \phi}{\partial x^a} = \sum_{b=0}^{3} \frac{\partial \phi}{\partial x'^b} \frac{\partial x'^b}{\partial x^a} \ . \tag{2}$$

A set of four quantities $A^a \equiv \{A^0, A^1, A^2, A^3\}$, is called a *vector with index up*[3] if it transforms as coordinate differentials in equation (1),

$$A^a = \sum_{b=0}^{3} \left(\frac{\partial x^a}{\partial x'^b} \right) A'^b \ . \tag{3}$$

A set of four quantities A_a is called a *vector with index down* if it transforms as derivatives of a scalar in equation (2),

$$A_a = \sum_{b=0}^{3} \left(\frac{\partial x'^b}{\partial x^a} \right) A'_b \ . \tag{4}$$

[1] For example, *time*, *latitude*, *longitude*, an *height* above the sea level.

[2] Indeed,

$$JJ^{-1} = \sum_{b=0}^{3} \frac{\partial x^a}{\partial x'^b} \frac{\partial x'^b}{\partial x^c} = \frac{\partial x^a}{\partial x^c} = \delta^a_c \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

[3] Also called contra-variant vector.

In the following we shall use the *implicit summation notation* where we drop the summation sign and always assume a summation over the index that appears in the up and down positions in a term,

$$\boxed{A^a B_a \equiv \sum_{a=0}^{3} A^a B_a \ .} \tag{5}$$

Quantities which transform as vectors or their products—that is, linearly through Jacobi matrix and its inverse—are said to transform *covariantly.* These quantities are called *tensors*. The number of the indices is called the *rank* of the tensor.

The *contraction* $A^a B_a$ is apparently a scalar (tensor with rank zero), as it is invariant under coordinate transformation. Indeed,

$$A^a B_a = \frac{\partial x^a}{\partial x'^b} A'^b \frac{\partial x'^c}{\partial x^a} B'_c = A'^b B'_b \ . \tag{6}$$

Metric tensor The *metric tensor* g_{ab} defines the metric – the invariant infinitesimal interval, ds^2, between two close events in time-space separated by dx^a,

$$ds^2 = g_{ab} dx^a dx^b \ . \tag{7}$$

Since dx^a is an arbitrary index-up vector, the construction $g_{ab} dx^b$ transforms as an index-down vector and thus the metric tensor connects index-up and index-down vectors,

$$A_a = g_{ab} A^b \ . \tag{8}$$

Covariant differential In curvilinear coordinates the ordinary differential of a vector, dA^a, is not a covariant quantity, since generally

$$dA_a = d(g_{ab} A^b) = g_{ab} dA^b + dg_{ab} A^b \neq g_{ab} dA^b \ , \tag{9}$$

if $dg_{ab} \neq 0$, that is, if the metric tensor is not constant throughout time-space.

In order to write covariant differential equations we need a *covariant differential*, denoted as DA^a, which satisfies the covariance condition,

$$DA_a = g_{ab} DA^b = Dg_{ab} A^b. \tag{10}$$

The covariant differential has to contain dA^a and an additional term which would ensure covariance. As a contribution to differential the additional term must be linear in dx. Moreover, it must also be linear in A if we demand linearity of the operation of differentiation. The covariant differentical can then be generally written as

$$DA^a = dA^a + \Gamma^a_{bc} A^b dx^c \ , \tag{11}$$

where the factor Γ^a_{bc} (apparently, not itself a covariant tensor) is called the *Christoffel symbol.*

Differential of a scalar $d(A_a B^a)$ is already a covariant quantity, therefore

$$D(A_a B^a) = d(A_a B^a) \tag{12}$$

for an arbitrary B^a. If we demand that covariant differential satisfies the Leibnitz rule[4] we find[5]

$$DA_a = dA_a - \Gamma^b_{ac} A_b dx^c \ . \tag{13}$$

[4] The Leibnitz rule states that

$$D(AB) = A\,DB + B\,DA \ .$$

[5] Applying the Leibnitz rule to (12) gives

$$DA_a B^a + A_a (dB^a + \Gamma^a_{bc} B^b dx^c) = dA_a B^a + A_a dB^a \ .$$

Simplifying and exchanging $a \leftrightarrow b$ in the Γ-term gives

$$DA_a B^a = (dA_a - \Gamma^b_{ac} A_b dx^c) B^a \ .$$

Since B^a is arbitrary, $DA_a = dA_a - \Gamma^b_{ac} A_b dx^c$.

With the comma-notation for partial derivatives,

$$dA^a = \frac{\partial A^a}{\partial x^c} dx^c \equiv A^a_{,c} dx^c \; , \tag{14}$$

the covariant differentials take the form

$$DA^a = \left(A^a_{,c} + \Gamma^a_{bc} A^b\right) dx^c \; , \tag{15}$$

$$DA_a = \left(A_{a,c} - \Gamma^b_{ac} A_b\right) dx^c \; . \tag{16}$$

The expressions in the parentheses are tensors, since when multiplied by arbitrary vector dx^c they produce vectors. These tensors are called *covariant derivatives* and are denoted as

$$A^a_{;c} = A^a_{,c} + \Gamma^a_{bc} A^b \; , \tag{17}$$

$$A_{a;c} = A_{a,c} - \Gamma^b_{ac} A_b \; . \tag{18}$$

Christoffel symbols and the metric tensor Although the Christoffel symbol itself is not a tensor, the linear combination

$$\Gamma^a_{bc} - \Gamma^a_{cb} \tag{19}$$

is a tensor. Indeed, consider the difference,

$$\varphi_{;c;b} - \varphi_{;b;c} = (\Gamma^a_{bc} - \Gamma^a_{cb})\varphi_{;a} \; , \tag{20}$$

where φ is an arbitrary scalar. The left-hand side is a tensor and $\varphi_{;a}$ is also a tensor, therefore $\Gamma^a_{bc} - \Gamma^a_{cb}$ is also a tensor. The latter is equal zero in a locally flat frame. Being a tensor it is than equal zero in all other frames. Therefore Christoffel symbol is symmetric[6] over exchange of the two lower indexes,

$$\Gamma^a_{bc} = \Gamma^a_{cb} \; . \tag{21}$$

The covariance condition (10) is fullfilled if the covariant derivative of the metric tensor[7] vanishes,

$$Dg_{ab} = 0 \; . \tag{22}$$

This condition together with the symmetry (21) determines[8] the Christoffel symbol through the derivatives of the metric tensor,

$$\Gamma_{abc} = \frac{1}{2}\left(g_{ab,c} - g_{bc,a} + g_{ac,b}\right) . \tag{23}$$

[6] Spaces with this property are called *torsion-free*.

[7] The Leibnitz rule $D(A^a B^b) = DA^a B^b + A^a DB^b$ gives the definition of the covariant differential of a tensor as

$$DF^{ab} = dF^{ab} + \Gamma^a_{cd} F^{cb} dx^d + \Gamma^b_{cd} F^{ac} dx^d .$$

[8] The (vanishing) Dg_{ab} is given as

$$Dg_{ab} = dg_{ab} - \Gamma^d_{ac} g_{db} dx^c - \Gamma^d_{bc} g_{ad} dx^c = 0 \; .$$

This can be written as

$$g_{ab,c} - \Gamma_{bac} - \Gamma_{abc} = 0 \; .$$

Exchanging indices gives two other equations,

$$-g_{bc,a} + \Gamma_{cba} + \Gamma_{bca} = 0 \; ,$$

$$g_{ca,b} - \Gamma_{acb} - \Gamma_{cab} = 0 \; .$$

Finally, adding the last three equations gives (23).

Exercises

1. **Covariant differentiation in polar coordinates**

   Let x, y be Cartesian coordinates in a flat two-dimensional space with the metric

   $$dl^2 = dx^2 + dy^2 \ .$$

   Consider the polar coordinates

   $$x = r\cos\theta \ , \quad y = r\sin\theta \ .$$

   (a) Calculate the interval dl^2 in polar coordinates and find the corresponding metric tensor g_{ab} (the indexes a and b in polar coordinates take the values r, θ).

   (b) The line element $d\vec{l}$ is defined as

   $$d\vec{l} = dx\vec{e}_x + dy\vec{e}_y \ ,$$

   where $\vec{e}_x$ and $\vec{e}_x$ are the Cartesian uint vectors,

   $$\vec{e}_x \cdot \vec{e}_x = \vec{e}_y \cdot \vec{e}_y = 1 \ , \quad \vec{e}_x \cdot \vec{e}_y = 0 \ .$$

   Consider this line element in polar coordinates,

   $$d\vec{l} = dr\vec{e}_r + d\theta\vec{e}_\theta \ ,$$

   and find the polar unit vectors $\vec{e}_a$, $a = r, \theta$.

   (c) Argue and check that $\vec{e}_a \cdot \vec{e}_b = g_{ab}$.

   (d) Find $g^{ab} = g^{-1}_{ab}$, $\vec{e}^a = g^{ab}\vec{e}_b$, and $\vec{e}^a \cdot \vec{e}_b$.

   (e) Consider a vector $\vec{A} = A^a\vec{e}_a = A_a\vec{e}^a$. Consider the relation between the actual differential of the vector, $DA^a = \vec{e}^a \cdot d\vec{A}$, and the apparent differential of the vector, $dA^a = d(\vec{e}^a\vec{A})$: prove that[9]

   $$DA^a = dA^a + (\vec{e}^a \cdot \vec{e}_{b,c})\, A^b dx^c,$$

   $$DA_a = dA_a - \left(\vec{e}^b \cdot \vec{e}_{a,c}\right) A_b dx^c.$$

   Argue that the expression in parentheses is the Christoffel symbol.

2. Argue, that the Christoffel symbol is not a tensor.

3. A traveller starts from Earth and moves along a line with constant acceleration g for 25 traveller's years then with constant deceleration g again for 25 traveller's years. How far from Earth will they reach? What was their maximum speed in the Earth's frame (assumed inertial)? The traveller then flies back to Earth in the same manner. How many years will have passed on Earth since his departure when he comes back to Earth?

[9] comma-index denotes partial derivative,

$$\phi_{,c} \equiv \frac{\partial\phi}{\partial x^c} \ .$$