## Equivalence principle

General relativity is based on the Einstein's equivalence principle, which postulates local equivalence between inertial and gravitational forces.

### Inertial forces

Inertial forces are apparent forces that act on all bodies in a non-inertial frame of reference, such as a rotating frame (centrifugal and Coriolis forces) or a uniformly accelerating frame (elevator force). Inertial forces do not arise from any physical interaction but rather from the non-linear motion of the frame itself.

In an inertial frame a free body moves without acceleration, and its (non-relativistic) equation of motion in Cartesian coordinates is

$$\ddot{\vec{r}} = 0, \tag{1}$$

where $\vec{r} \doteq \{x, y, z\}$ are the three spatial coordinates of the body and dots denote time derivative. This is an equation for a linear motion – in inertial frames free bodies move without acceleration along straight curves.

In a non-inertial uniformly accelerating frame, which moves with acceleration $\vec{a}$ relative the to an inertial frame, the corresponding equation of motion for the free body is

$$\ddot{\vec{r}} = -\vec{a}. \tag{2}$$

This is an equation for a (not straight) curve (a parabola, actually) – in non-inertial frames free bodies generally move with acceleration along curves.

The system of coordinates, where free bodies move along curves, rather than lines, are called *curvilinear*.

Equation (2) can be written in the form of the second Newton's law,

$$m\ddot{\vec{r}} = \vec{F}_I, \tag{3}$$

where $m$ is the mass of the body, and

$$\vec{F}_I = -m\vec{a} \tag{4}$$

is the inertial force (in this case often called *elevator force*).

Inertial forces have the following properties:

1. Under inertial forces all bodies move with the same acceleration, that is, inertial forces are proportional to the masses of the bodies.
2. Inertial forces arise from geometrical properties of curvilinear coordinate systems rather than due to some physical fields affecting the bodies.
3. Inertial forces disappear after a coordinate transformation to an inertial frame.

### Newton's gravitational forces

Newton's law of gravitation states that the gravitational force $\vec{F}_G$ acting on a particle from some gravitating bodies is proportional to the mass $m$ of the particle,

$$\vec{F}_G = m\vec{g}\ , \tag{5}$$

where the gravitational acceleration $\vec{g}$ depends on the positions and masses of the gravitating bodies. Therefore under gravitational forces all bodies move with the same acceleration as the mass disappears from the second Newton's law of motion,

$$m\ddot{\vec{r}} = m\vec{g}\ . \tag{6}$$

The masses in the left- and right-hand sides of this equation are often referred to separately as inertial and gravitational masses.

Many experiments attempted to observe the difference between gravitational and inertial masses, including Galileo's measurements of acceleration of balls of different composition rolling down inclined planes, and Newton's measurements of the period of pendulums with different mass but identical length. All experiments reported no observed difference, with the most precise experiment to day (arXiv:0712.0607) having the accuracy of one part in $10^{13}$.

These experiments suggest that all bodies under gravitational forces move with the same acceleration, just like under inertial forces.

### Einstein's equivalence principle

Einstein has postulated that gravitational forces are locally[1] equivalent to inertial forces, that is, gravitational forces are unlike other physical forces but much like geometrical inertial forces. This postulate is called the Einstein's equivalence principle. It can be formulated in several ways:

1. Gravitational forces are locally equivalent to inertial forces.
2. An accelerated frame is locally equivalent to a frame in a uniform gravitational field.
3. Gravitational field is locally equivalent to a non-inertial frame.
4. In free fall the effects of gravity disappear in all possible local experiments and general relativity reduces locally to special relativity.

The principle is customarily illustrated by two spacecrafts, one on Earth, the other accelerating with the Earth's gravitational acceleration $g$ in the outer space. The observers in these spacecrafts can not determine by doing local experiments inside spacecrafts whether their craft is accelerating or at rest in a gravitational field or accelerating.

Unlike inertial forces, gravitational forces vanish at large distances from the sources of gravitation and therefore non-local experiments can well distinguish between fictitious and gravitational forces.

## Time-space in a gravitational field

### Flat and curved spaces

A space with Euclidean or pseudo-Euclidean metric is called *flat*. For example, the Minkowski space of special relativity is flat.

The coordinates in which the metric is (pseudo)-Euclidean are also often called flat. If the metric is not everywhere (pseudo)-Euclidean, the coordinates are called *curvilinear*. For example, in a flat two-dimensional space with polar coordinates $\{r, \theta\}$ the metric is non-Euclidean,

$$dl^2 = dr^2 + r^2 d\theta^2. \tag{7}$$

However, if there exist a coordinate transformation that globally turns the metric into (pseudo)-Euclidean, the space is still called flat. In our example such transformation is

$$\begin{cases} x = r\cos\theta\ , \\ y = r\sin\theta\ . \end{cases} \tag{8}$$

If such transformation does not exist, the space is called *curved* or *Riemann space*. The geometry of a curved space is called non-Euclidean geometry or *Riemann geometry*.

[1] *locally* means within a limited space, where variations of the fields can be neglected.

### Space-time in a gravitational field

Einstein's equivalence principle states that gravitational forces disappear locally in the frame of a free falling observer. A free falling observer finds themselves in a local Minkowski space-time.

In other words the space-time in a gravitational field can be transformed locally to Minkowski space-time by a non-linear coordinate transformation.

Yet in other words the space-time in a gravitational field locally can be obtained by a non-linear coordinate transformation from a Minkowski space-time.

This transformation cannot be global though since gravitational forces, unlike inertial forces, vanish at large distances from massive bodies.

Therefore the consequence of Einstein's equivalence principle is that the space-time in a gravitational field is genuinley curved.

### General principle of relativity

In the presence of gravitational fields the space-time is curved, and it is not possible to build a set of globally flat coordinates. Therefore any frame of reference with arbitrary curvilinear coordinates and arbitrarily tuned clocks must be equally accepted in general relativity[2].

The general principle of relativity can then formulated as

*The laws of physics should have the same form in arbitrary frames of reference.*

In order to build suitable differential equations, invariant under general coordinate transformations, we need to develop differential geometry in curvilinear coordinates.

### Exercises

1. A pendulum is suspended from the roof of a car moving in a line (straight curve) with constant acceleration $\vec{a}$. Find the angle the pendulum makes with the vertical. Explain what is happening from the viewpoint of an intertial observer outside of the car and a non-inertial observer in the car.

2. A bucket of water slides freely under gravity down a slope of a fixed angle $\alpha$ to the horisontal. What is the angle of inclination of the surface of whater relative to the base of the bucket?

3. Argue that the equation of motion of a test body with coordinate $\vec{r}$ in Newton's theory of gravitation,
$$\ddot{\vec{r}} = -\sum_k \frac{G_N M_k}{|\vec{r}-\vec{r}_k|^2} \frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|} \,, \tag{9}$$
where $M_k$ and $\vec{r}_k$ are the masses and coordinates of gravitating bodies, can be written as
$$\ddot{\vec{r}} = -\vec{\nabla}\phi \,, \tag{10}$$
where $\phi$ is the gravitational potential,
$$\phi(\vec{r}) = -\sum_k \frac{G_N M_k}{|\vec{r}-\vec{r}_k|} \,. \tag{11}$$
Argue that the graviational potential satisfies the equation[3].
$$\nabla^2\phi(\vec{r}) = 4\pi G_N \mu(\vec{r}) \,, \tag{12}$$

[2] The accepted coordinates must be differentiable (to allow development of differential geometry) and the coordinate transformations must be invertible.

[3] Hint: $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$.

where $\mu(\vec{r})$ is the mass density of gravitating bodies,

$$\mu(\vec{r}) = \sum_k M_k \delta(\vec{r} - \vec{r}_k) \; . \tag{13}$$

Argue that the equation of motion (9) and the equation for the gravitational potential (12) are invariant under Galilean transformation.

4. Argue that the non-relativistic motion of a test body with mass $m$ in a Newtonian gravitational potential $\phi$ can be described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} m v^2 - m\phi - mc^2 \; , \tag{14}$$

where $v$ is the velocity of the body, whith the corresponding action

$$S = \int dt \left( \frac{1}{2} m v^2 - m\phi - mc^2 \right) = -mc \int c dt \left( 1 - \frac{v^2}{2c^2} + \frac{\phi}{c^2} \right) . \tag{15}$$

Argue (by comparing with $S = -mc \int ds$) that the infinitesimal interval for the body is then given as

$$ds^2 = \left( 1 + \frac{2\phi}{c^2} \right) c^2 dt^2 - d\vec{r}^2 \; . \tag{16}$$

Argue that this metric describes a curved space and that the proper time runs differently at different levels of the gravitational potential.

5. Explane the "twins paradox" of special relativity from the viewpoints of the inertial twin and the non-inertial (travelling) twin.