Motion in the Schwarzschild metric

Motion in the Schwarzschild metric reveals several of the unusual consequences of general relativity:

- utmost relativity of measurements: it takes finite proper time for a body to fall onto the Schwarzschild radius, yet for an outside observer it takes infinite time;
- there exist singularities (geodesic incompleteness) in general relativity: some trajectories cannot be extended beyond a certain point;
- there exist event horizons in general relativity — the hyper-surfaces in time-space which can only be crossed in one direction.

Lemaitre coordinates

In the Schwarzschild metric around a body with the gravitational radius r_g

$$ds^2 = (1 - \frac{r_g}{r})dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \quad (1)$$

there is a singularity at the gravitational radius, $r = r_g$. Under the gravitational radius the coordinate r becomes time-like and t becomes space-like.

However, it turns out to be not a physical singularity, but rather an artifact of the (incorrect) assumption that a static Schwarzschild coordinates can be realized under the gravitational radius with material bodies. Such removable singularities are called *coordinate singularities*.

A transformation to the Lemaitre coordinates τ, ρ

$$d\tau = dt + \sqrt{\frac{r_g}{r}} \frac{dr}{1 - \frac{r_g}{r}} , \quad (2)$$

$$d\rho = dt + \sqrt{\frac{r}{r_g}} \frac{dr}{1 - \frac{r_g}{r}} , \quad (3)$$

leads to the Lemaitre metric, where the singularity at r_g is removed[1],

$$ds^2 = d\tau^2 - \frac{r_g}{r} d\rho^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) , \quad (4)$$

where $r = [\frac{3}{2}(\rho - \tau)]^{2/3} r_g^{1/3}$. The latter is obtained by integrating

$$d\rho - d\tau = \sqrt{\frac{r}{r_g}} dr , \quad (5)$$

which is the difference between (3) and (2).

The Lemaitre coordinates are synchronous[2] and are thus realized by a system of clocks in a free radial fall from infinity towards the origin.

Radial fall towards the origin

For a free falling body $d\rho = 0$ and equation (3) gives

$$dt = -\sqrt{\frac{r}{r_g}} \frac{1}{(1 - \frac{r_g}{r})} dr . \quad (6)$$

Approaching the Schwarzschild radius, in the region $r \gtrsim r_g$, we have in the lowest order in $(r - r_g)/r_g$,

$$dt = -\frac{r_g}{r - r_g} dr , \quad \Rightarrow \quad \frac{r - r_g}{r_0 - r_g} = e^{-\frac{t - t_0}{r_g}} . \quad (7)$$

[1] there remains a genuine singularity at the origin.
[2] the metric has the form $ds^2 = d\tau^2 + g_{\alpha\beta} dx^\alpha dx^\beta$.

Apparently, it takes a free falling body infinitely long t-time — the time used by the outer observer — to reach the Schwarzschild radius.

On the contrary, a free falling Lemaitre clock moves from some radius r_1 to a smaller radius r_2 — which can well be the gravitational radius or even the origin — within finite τ-time $\Delta\tau_{12}$. Indeed, setting $d\rho = 0$ in (5) gives

$$\Delta\tau_{12} = -\int_{r_1}^{r_2} \sqrt{\frac{r}{r_g}} dr = \frac{2}{3}\left(\frac{r_1^{3/2} - r_2^{3/2}}{r_g^{1/2}}\right) . \tag{8}$$

Event horizons and black holes

Along the trajectory of a radial light ray

$$ds^2 = d\tau^2 - \frac{r_g}{r} d\rho^2 = 0 , \tag{9}$$

which gives

$$d\rho = \pm\sqrt{\frac{r}{r_g}} d\tau , \tag{10}$$

where plus and minus describe the rays of light sent correspondingly up and down.

Isolating $d\rho$ in (5) and inserting the result into (10) shows that along the trajectory

$$dr = \left(\pm 1 - \sqrt{\frac{r_g}{r}}\right) d\tau . \tag{11}$$

Apparently if $r < r_g$ then there is always $dr < 0$ and thus the light rays emitted radially inwards and outwards both end up at the origin. In other words no signal can escape from inside the gravitational radius. This phenomenon is called the event horizon.

Therefore a massive object with a size less than the gravitational radius, called a black hole, is completely under the event horizon and its interior is totally invisible.

The trajectories of massive bodies and light rays inside the gravitational radius both end up in the origin where they cannot be extended any further.

The black holes can possibly be detected through their interaction with the matter outside the event horizon.

Exercises

1. What is the escape velocity[3] at a coordinate r in the Schwarzschild field? Investigate the limits $r \gg r_g$ and $r \to r_g$.

   (a) argue that the equation of motion of a free radially moving body in the Schwazschild field can be written as

   $$\left(1 - \frac{r_g}{r}\right)\frac{dt}{ds} = E \ , \quad E^2 - \left(\frac{dr}{ds}\right)^2 = 1 - \frac{r_g}{r}.$$

   (b) argue that if the body has reached $r = \infty$ the integration constant E is its energy divided by mass;

   (c) argue that for a body, starting a free radial fall from infinity with zero velocity, $E = 1$.

[3] argue that a body starting a free radial fall at infinity with zero velocity will reach a given r with $\frac{dr}{ds}$ equal escape velocity.