Prologue

General relativity is a classical relativistic theory of gravitation published by Albert Einstein in 1916. It is the accepted description of gravitation in modern physics.

General relativity is a geometric theory where gravitational field is not a material field but rather a curvature of space-time: massive bodies distort space-time in their vicinity which affects the motion of other bodies.

General relativity satisfies the correspondence principle[1]: in the absence of gravitational fields general relativity reduces to special relativity; and in the limit of weak gravitational fields and non-relativistic velocities it reduces to Newtonian gravitation.

Although not the only relativistic theory of gravitation, general relativity is the simplest theory consistent with experimental data.

General relativity has important astrophysical implications and is a basis of current cosmological models of the universe.

Unlike classical electrodynamics general relativity has not been quantized – a complete and self-consistent theory of quantum gravity does not exist yet.

Special relativity

Special relativity is a theory of spatial and temporal measurements in inertial frames of reference, and of relativistic kinematics. It was formulated by Albert Einstein in 1905. Special relativity is the basis of relativistic mechanics. In the slow motion limit special relativity reduces to Galilean relativity.

Postulates

Special relativity is based on several postulates[2],

1. **Homogeneity and isotropy of space**: The space is homogenious and isotropic.
2. **Existence of inertial frames**: (In the absence of gravitational forces) there exist inertial frames of reference with Cartesian coordinates where the laws of physics take their simplest form. In particular, free bodies—that is, bodies not affected by forces—move with constant velocities along lines (straight curves), from which it follows, that the coordinates in different inertial frames are connected by a linear transformation. Inertial frames move with constant velocities with respect to each other.
   Exercise: argue that coordinate transformations between inertial frames form a mathematical group[3].
3. **Existence of finite maximum speed**: there exist a finite maximum speed (which actually relatively small, 299792458m/s) with which a physical object can travel relative to a physical observer.

   Equivalently, one can rather postulate—as Einstein originally did—the **constancy of the speed of light**, as motivated by Maxwell's theory of electromagnetism and the null result of the Michelson–Morley experiment.
4. **Special principle of relativity**: all inertial frames are equivalent and the laws of physics have the same form in all inertial frames.

[1] The correspondece princeiple suggests that a new theory should reproduce the results of older well-established theories in those domains where the old theories work.

[2] In physics, a *postulate* is a physical law of a more general nature which is typically deduced from a large number of different experiments.

[3] In mathematics, a *group* is a set of elements together with an operation that combines any two of its elements to form a third element also in the set while satisfying four conditions called the group axioms, namely *closure*, *associativity*, *identity* and *invertibility*.

Lorentz transformation

Lorentz transformation relates the measurments of spatial and temporal intervals in different inertial frames. It is a linear transformation: it transforms a linear motion of a free body in one inertial frame to an equally linear motion of the the same body in another frame.

Let us consider a linear transformation of coordinates between two inertial frames with parallel Cartesian coordinates moving with relative velocity v along one of the axes[4]. The general form of such transformation, consistent with isotropy of the space and the group postualtes, has the form

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{1}$$

where (t', x') are the coordinates in the frame K' which moves relative to the frame K with coordinates (t, x) with velocity v along the x (and x') axis. The y- and z-coordinates, perpendicular to the velocity boost, transform identically and are therefore omitted for brevity.

The velocity c is a universal constant, the fastest possible relative velocity of two inertial frames. It is experimentally measured to be finite and to be equal the speed of light (in vacuum).

Transformation (1) with finite c is called the Lorentz transformation. Note that time and space do not transform separately but rather as components of one inseparable four-component space-time point (t, x, y, z).

In the limit $c \to \infty$ the Lorentz transformation turns into Galilean transformation,

$$\begin{aligned} t' &= t\ , \\ x' &= x - vt\ . \end{aligned} \tag{2}$$

Here time is absolute and does not transform at all. The time-space coordinates then separate into invariant time and three spatial coordinates.

Invariant interval and metric

A direct calculation shows that the infinitesimal *interval*,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2\ , \tag{3}$$

is invariant under the Lorentz transformation (1). It thus defines a *metric*[5]. A space with a metric is called *metric space*.

The pseudo[6]-Euclidean metric (3) is called *Minkowski metric* and a space with such metric is called *Minkowski space*.

The existence of a metric allows developement of a geometry of space: measurements of distances, angles, and time intervals. However, geometry in Minkowski space is sometimes different from the everyday Euclidean geometry. In particular, distances and time intervals are relative, that is, they are different in different frames.

In the limit $v \ll c$ Minkowski space reduces to *Euclidean space*, which is the non-relativistic world of classical mechanics with Galilean transformation where dt is itself invariant and the Minkowski metric reduces to the Euclidean metric,

$$dl^2 = dx^2 + dy^2 + dz^2\ . \tag{4}$$

[4]This transformation is often called *Lorentz boost*, or *velocity boost*, or simply *boost*.

[5]A *metric* is a function that defines a distance between two infinitesimally close points in a space. Metric is used to measure distances and angles and thus to develop a geometry of the space.

[6]*Euclidean metric* in an n-dimensional space has the form

$$ds^2 = dx_1^2 + \cdots + dx_n^2,$$

while *pseudo-Euclidean metric* has one or more negative signs,

$$ds^2 = dx_1^2 + \cdots + dx_k^2 - dx_{k+1}^2 - \cdots - dx_n^2.$$

Relativistic kinematics

The postulate that free bodies move along lines can be conveniently formulated through the variational (least action) principle. Indeed a line between two points is the curve with extremal measure. The measure of a curve in a metric space is given by the integral

$$\int ds \tag{5}$$

taken along the curve. The free bodies thus move along curves with extremal measure or, equivalently, along curves with vanishing variation of the measure,

$$\delta \int ds = 0 \; . \tag{6}$$

The postulate about the motion of free bodies can then be reformulated as a stationary action principle with the action

$$\mathcal{S} = \alpha \int ds \; , \tag{7}$$

where the constant α can be deduced from the correspondence principle: in the non-relativistic limit the action of a free body has to take the classical form, namely the kinetic energy of the body,

$$\mathcal{S} \xrightarrow{v \ll c} \int dt \frac{mv^2}{2} \; . \tag{8}$$

Calculating the nonrelativistic limit of (7),

$$\mathcal{S} = \alpha \int c dt \sqrt{1 - \frac{v^2}{c^2}} \xrightarrow{v \ll c} \alpha c \int dt \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) \; , \tag{9}$$

and comparing with (8) gives $\alpha = -mc$,

$$\mathcal{S} = -mc \int ds = -mc \int \sqrt{c^2 dt^2 - d\vec{r}^2} = -mc^2 \int dt \sqrt{1 - \frac{\vec{v}^2}{c^2}} \; . \tag{10}$$

The Lagrangian $\mathcal{L}$ of a free body is thus given as

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}} \; . \tag{11}$$

Having the Lagrangian one can obtain in the usual way the momentum $\vec{p}$,

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \; , \tag{12}$$

and the energy $\mathcal{E}$,

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \vec{v}}\vec{v} - \mathcal{L} = \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \; , \tag{13}$$

of the body.

Exercises

1. Show that the action of a body in the form

$$\mathcal{S} = \int \mathcal{L}(\vec{r}, \vec{v}) dt \tag{14}$$

leads—through the variational principle which demands $\delta\mathcal{S} = 0$ on the trajectory—to the (Euler-Lagrange) equation of motion,

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\vec{v}} = \frac{\partial\mathcal{L}}{\partial\vec{r}}, \tag{15}$$

for the trajectory of the body.

2. Argue that a free body with action $\mathcal{S} = -mc\int ds$ moves along a line (a straight curve).

3. Momentum $\vec{p}$ is the quantity which conserves (along the trajectory of the body) if the Lagrangian does not depend explicitely on $\vec{r}$ (through the Noether's Theorem). Argue, that

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\vec{v}}. \tag{16}$$

4. Energy $\mathcal{E}$ is the quantity which conserves (along the trajectory of the body) if the Lagrangian does not depend explicitely on time (through the Noether's Theorem). Indeed in this case the variation of the Lagrangian under the infinitesimal transformation $t \to t + dt$ is given as

$$d\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\vec{r}}d\vec{r} + \frac{\partial\mathcal{L}}{\partial\vec{v}}d\vec{v}\ . \tag{17}$$

Show that on the trajectory this can be written as the energy conservation law,

$$\frac{d\mathcal{E}}{dt} = 0\ , \tag{18}$$

with the energy

$$\mathcal{E} = \frac{\partial\mathcal{L}}{\partial\vec{v}}\vec{v} - \mathcal{L}\ . \tag{19}$$

5. Consider the motion of a particle with charge e and mass m in a constant uniform electric field $\vec{E}$ which is, say, in the direction of the x-axis.

   (a) Suppose that at $t = 0$ the particle was at rest, $\vec{v} = 0$, with the coordinate $\vec{r} = 0$. Find $x(t)$.

   (b) Suppose that at $t = 0$ the particle had $\vec{r} = 0$ and $v_x = 0$, but $v_y \neq 0$. Find $x(t)$, $y(t)$ and $x(y)$.

   (c) Consider the limits $eEt \ll mc$ and $eEt \gg mc$.

Hint: the equation of motion of a charged particle in an electro-magnetic field $\vec{E}$, $\vec{H}$ is

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{H}\right)\ , \tag{20}$$

where the (relativistic) momentum $\vec{p}$ and the velocity $\vec{v}$ are related as

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}\ . \tag{21}$$

6. Derive the Lorentz transformation one way or another, for example:

   (a) from isotropy of space, group postulates, and finite maximum velocity;

   (b) from isotropy of space and invariance of the speed of light;

   (c) the way it was done in your textbook of special relativity;

   (d) any other way.

7. Show that in Minkowski space the finite interval, $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$, is also invariant.