General Relativity. Examination problems.
Fall 2015.

For instructions see the course homepage at

`http://owww.phys.au.dk/~fedorov/GTR`

1 Part 1 (of 4)

1. Let Λ^a_b be the matrix of the Lorentz transformation from an inertial frame K to an intertial frame K'. What is the matrix of the Lorentz transformation from K' to K?
2. Two vectors A and B have equal components, $A^a = B^a$, in a given frame. Do they have equal components in other frames? Give the answers for special and general relativity.
3. Two close events in special relativity are separated by i) a time-like interval, $ds^2 > 0$, ii) a space-like interval, $ds^2 < 0$. Is there a frame where the two events are I) simultaneous, II) happen at the same spatial point?
4. Consider a 3-dimensional Euclidean space with polar coordinates. Is there a difference between vectors with index up and vectors with index down?
5. Consider a tensor which is antisymmetric, $F_{ab} = -F_{ba}$, in a given frame. Is it antisymmetric in other frames? What can one say about the symmetry of the tensor F^{ab} (that is, with indexes up)?
6. A vector has zero components in one frame. Are the components zero in *all* other frames? Give the answers for special and general relativity.
7. Suppose that the relative velocity v between two inertial frames is close to the speed of light, $v = c - \varepsilon$, where $0 < \varepsilon \ll c$. Derive the formulas for the Lorentz contraction and time-dilation in the lowest order in ε. What are the relative errors of these formulas when $\varepsilon = 0.1c$?
8. Argue that the space of special relativity is a metric space with the metric tensor η_{ab} (a diagonal tensor where the components of the main diagonal are equal $(1, -1, -1, -1)$).
9. In special relativity: argue that a free electron can neither absorb nor emit a photon.
10. Consider a 3-dimensional Euclidean space with Cartesian coordinates. Is there a difference between vectors with index up and index down?
11. Consider a vector A^a. Is the four-component object $\left\{\frac{1}{A^0}, \frac{1}{A^1}, \frac{1}{A^2}, \frac{1}{A^3}\right\}$ a vector?
12. Two close events in special relativity are separated by a zero interval, $ds^2 = 0$. What can be said about the spatial and temporal separation of the two events?
13. Consider a tensor $X^a_{\ bc}$. Is the quantity $Y_c = X^a_{\ ac}$ a tensor?
14. Consider a tensor which is symmetric, $S_{ab} = S_{ba}$, in a given frame. Is it symmetric in other frames? What can one say about the symmetry of the tensor S^{ab} (with indexes up)?

15. Calculate the energy that is required to accelerate a particle with mass $m > 0$ from speed v to speed $v + \delta v$ where $\delta v \ll v$. Argue that it would take an infinite amount of energy to accelerate the particle to the speed of light.

2 Part 2 (of 4)

1. In some inertial frame the motion of a particle is described by the equations

$$x(t) = at + b\sin(\omega t),\ y(t) = b\cos(\omega t),\ z(t) = 0,\ |b\omega| < 1\,.$$

Compute the components of particle's four-velocity and four-acceleration.

2. Calculate the Riemann tensor for a cylinder. You are free to choose the system of coordinates.

3. In a two-dimensional Euclidean space with polar coordinates consider a tensor A^{ab} with the following components: $A^{rr} = r^2, A^{r\theta} = r\sin\theta, A^{\theta r} = r\cos\theta, A^{\theta\theta} = \tan\theta$. Compute the covariant derivatives $A^{ab}_{;c}$ (in polar coordinates).

4. Differentiate a determinant of a 2×2 matrix and show that it satisfies the equation $g_{,c} = gg^{ab}g_{ba,c}$.

5. Consider a space with coordinates (u, v, w, p) and metric $ds^2 = 2dudv - dwdw - dpdp$. Show that this is Minkowski space.

6. In special relativity: consider the 4-velocity $u^a = dx^a/ds$ and the 3-velocity $\vec{v} = d\vec{r}/dt$ of a moving body with coordinates $x^a = \{t, \vec{r}\}$. Express u^a in terms of $\vec{v}$.

7. Calculate the components of g^{ab} for a sphere in polar coordinates.

8. In a two-dimensional Euclidean space with polar coordinates the components of a vector field are given as $A^r = 1$, $A^\theta = 0$. Compute $A^a{}_{;b;c}$.

9. Prove that $\Gamma^a{}_{ab} = \frac{1}{2}(\ln|g|)_{,b}$.

10. Find out how the expression $A^a{}_{,b}$ transforms under a change of coordinates. Does it obey the tensor transformation law?

11. Show that the Christoffel symbol Γ^a_{bc} is symmetric under exchange of the lower indices.

12. Consider a scalar function of coordinates $\phi(x)$. Find out whether the objects $\frac{\partial\phi}{\partial x^a}$ and $\frac{\partial^2\phi}{\partial x^a \partial x^b}$ are tensors in special and general relativity.

13. Prove that in a locally inertial frame

$$R_{abcd,e} = \frac{1}{2}(g_{ad,bce} - g_{ac,bde} + g_{bc,ade} - g_{bd,ace})\,.$$

14. Starting with Minkowski metric $ds^2 = \eta_{ab}dx^a dx^b$, show that the coordinate transformation $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arccos(z/r)$, $\varphi = \arctan(y/x)$ leads to metric $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$.

3 Part 3 (of 4)

1. A traveller starts from rest at the Earth and moves along a line with constant acceleration a with respect to the momentarily co-moving inertial frame (also called the instantaneous rest frame). Argue that light signals sent from the Earth after time $t = c/a$ will never reach the receding traveller.

2. The non-relativistic Lagrangian of a free particle with mass m is given as $L = \frac{1}{2}m\vec{v}^2$, where $\vec{v}$ is the particle's velocity. What is the corresponding relativistic Lagrangian? What is the connection between the non-relativistic and the relativistic Lagrangians?

3. The metric in a space is given as

$$ds^2 = (1+2\phi)dt^2 - (1-2\phi)(dx^2+dy^2+dz^2)\,,$$

where $|\phi| \ll 1$ everywhere. To first order in ϕ compute g^{ab}.

4. Prove that in a locally inertial frame

$$R_{abcd,e} = \frac{1}{2}(g_{ad,bce} - g_{ac,bde} + g_{bc,ade} - g_{bd,ace})\,.$$

5. Calculate the components of the Riemann tensor for a 2-dimensional Euclidean space with polar coordinates.

6. Assume that a geodesic is a curve with extremal measure, that is, a curve along which

$$\delta \int ds = 0\,.$$

Derive the geodesic equation. Prove that the covariant differential of the velocity of a particle moving along the geodesic is zero, $Du^a = 0$.

7. Calculate—using the metric—the length of a circle of constant coordinate θ on a sphere of radius r.

8. The metric in a space is given as

$$ds^2 = (1+2\phi)dt^2 - (1-2\phi)(dx^2+dy^2+dz^2)\,,$$

where $|\phi| \ll 1$ everywhere. To first order in ϕ compute the Christoffel symbols. Assume that ϕ is a function of (t, x, y, z).

9. Assume that the components of the metric tensor g_{ab} do not depend on the coordinate x^1. Show that for a free moving body the component u_1 of the body's four-velocity is then conserved.

10. Show that from the Maxwell equation $F^{ba}_{\ \ ;b} = 4\pi j^a$ it follows that $j^a_{\ ;a} = 0$. Hint: prove first that $\sqrt{-g}A^c_{\ ;c} = (\sqrt{-g}A^c)_{,c}$.

11. Use the definition of the invariant volume element (called covariant in lecture notes) to calculate the proper area of a sphere.

12. Show that for an equatorial orbit in the Schwarzschild metric the quantity u^2_φ is conserved (where u^φ is the $\varphi-$component of the four-velocity, $u^a = dx^a/ds$).

4 Part 4 (of 4)

1. In special relativity: Consider a massless scalar field Φ with the energy-momentum tensor

$$T_{ab} = \frac{1}{4\pi}\left(\Phi_{,a}\Phi_{,b} - \frac{1}{2}g_{ab}g^{cd}\Phi_{,c}\Phi_{,d}\right)\,.$$

From the equation $T^{ab}_{\ \ ,b} = 0$ derive the equations of motion for the field Φ. Now generalise your answer to general relativity.

2. The space is everywhere isotropic and empty (contains no matter at all). Show that it is Minkowski space. Hint: consider the Friedman equations.

3. In the weak field limit show that the metric tensor in the form $g_{ab} = \eta_{ab} + h_{ab}$, where $h_{ab\neq yz,zy} = 0$, $h_{yz} = f(t-x)$, and f is an arbitrary function, satisfies the linearized Einstein equation in vacuum. Hint: see the gravitational waves chapter.

4. Show that the equatorial orbit in the Schwarzschild metric is stable. Hint: consider an equatorial orbit with a small perturbation $\theta = \pi/2 + \delta\theta$; derive the lowest order equation for the perturbation $\delta\theta$; show that the perturbation remains small.

5. In Schwarzschild coordinates calculate the period of a circular orbit with radius b of a planet rotating around a star of mass M. Calculate also the proper period of the orbit. Compare with Newtonian result. Hint: recall the Kepler's law exercise.

6. The metric in a space is given as

$$ds^2 = (1+2\phi)dt^2 - (1-2\phi)(dx^2 + dy^2 + dz^2) \,,$$

where $|\phi| \ll 1$ everywhere. At a given point (t_0, x_0, y_0, z_0) find a coordinate transformation to a locally inertial frame, to first order in ϕ. At what rate does this frame accelerates with respect to the original coordinates, again to the first order in ϕ?

7. A rocket fell through the gravitational radius of a black hole and tries to escape back. Show that it will reach the center within the proper time $s \leq \pi M$ no matter how powerful the rocket engines are. Hints: argue that for a body under the Schwarzschild radius always $dr < 0$; from the condition $g_{ab}u^a u^b = 1$ prove the following inequality under the Schwarzschild radius,

$$\left(\frac{2M}{r} - 1\right)^{-1} \left(\frac{dr}{ds}\right)^2 > 1 \,,$$

from which you can obtain the limit on the proper velocity dr/ds; Maxima claims that

$$\int \frac{dr}{\sqrt{\frac{1}{r} - 1}} = -r\sqrt{\frac{1}{r} - 1} - \arctan\sqrt{\frac{1}{r} - 1} \,.$$

8. In Newtonian mechanics consider a planet rotating around a star which is slightly non-spherical, such that the classical Newtonian gravitational potential is

$$\phi(r) = -\frac{M}{r} - \frac{AM}{r^3} \,, \tag{1}$$

where the small parameter A describes the non-sphericity of the star. In Newtonian mechanics calculate the precession of the perihelion of the orbit of the planet. For simplicity you can assume that the orbit is nearly circular.

9. In the weak field limit show that the metric tensor in the form $g_{ab} = \eta_{ab} + h_{ab}$, where $h_{yz} = A\sin\omega(t-x)$, $h_{tt} = 2f(t-x)$, $h_{tx} = -f(t-x)$, all other $h_{ab} = 0$, f is an arbitrary function, satisfies the linearized Einstein equation in vacuum. Hint: see the gravitational waves chapter.

10. Let us define the generalized force $\mathcal{F}_a$ acting on a particle with mass m through the equation of motion $mDu_a/ds = \mathcal{F}_a$. Recall that from $u^a u_a = 1$ it follows that $Du_a u^a = 0$ and therefore $\mathcal{F}_a u^a = 0$. Show that the electromagnetic force satisfies this condition. Show that a force in the form $\mathcal{F}_a = -\partial V/\partial x^a$, where V is a scalar function of coordinates, generally does not satisfy this condition. What would a discovery of such force mean?

11. Consider a nearly circular orbit of a planet around a star in Newtonian mechanics and in General Relativity. Derive the equation for a small radial perturbation of the orbit and find its angular period. Relate to the post-Newtonian perihelion precession.
12. Consider a flat radiation-dominated Friedman universe. Show that at early times it expands as $a \propto \sqrt{t}$, where a is the scale parameter and t is coordinate time.
13. In Schwarzschild coordinates $\{t, r\}$ the half-life of an elementary particle measured at rest at large distance from the center is Δt. What is its half-life at rest at a distance r from the center? What is the proper half-life?