1. Argue that in special relativity $p^a = mu^a$, where $u^a \doteq dx^a/ds$ is the energy-momentum four-vector of a particle with mass m.

2. Derive the geodesic equation for the motion of a test body in a gravitational field—in the form with g_{ab} derivatives rather than Christoffel symbols—from the stationary action principle. Take the Schwarzschild metric as an example and write down the corresponding geodesics.

3. Argue that in the limit of a weak gravitational field the metric in vacuum can be written in the form

$$ds^2 = (1 + 2\varphi)dt^2 - (1 - 2\varphi)(dx^2 + dy^2 + dz^2),$$

where φ is the gravitational potential.

4. A body starts falling radially towards a black hole of mass M at the distance $r = \alpha R$, where $R = 2M$ and $\alpha > 1$, with zero initial velocity. Calculate the proper time it will take the body to reach i) the gravitational radius and ii) the centrum.

5. A body starts radial motion with zero initial velocity and constant acceleration a (away from the centrum) at the distance r_o from the centrum of a black hole of mass M. Find the acceleration a_o needed to keep the body at constant radius r_o. For $a < a_o$ and $r_o > 2M$ find the proper time it will take the body to reach the gravitational radius and the centrum.

6. An observer equipped with a radio-reciever starts falling radially towards a black hole of mass M at the distance $r = \alpha R$, where $R = 2M$ and $\alpha > 1$, with zero initial velocity leaving behind a radio-station wich transmits a short beep every second. The radio-station keeps constant radius $r = \alpha R$. How many beeps will the observer recieve until they reach the gravitational radius and the centrum?

7. Consider a nearly circular orbit of a planet around a star in Newtonian mechanics and General Relativity. Derive the equation for small radial oscillations of the orbit and find their angular period. Compare with the orbital period and relate to the post-Newtonian perihelion precession.

8. Gunnar Nordstrm suggested in 1913 a scalar geometric theory of gravitation where the metric tensor was postulated to have the form $g_{ab} = \psi^2\eta_{ab}$, where ψ is a scalar gravitational field and η_{ab} is the Minkowski metric tensor. Show that the theory satisfies the equivalence principle and the correspondence principle. Find the equation for the gravitational field. Argue that the theory does not bend the rays of light. Find the static spherically symmetric solution of the field equation. Find the precession of Mercury's perihelion in this theory.

9. Calculate the Newtonian gravitational potential of the Sun at the radius of the Earth's orbit and the potential of the Earth at its surface. Why do we very much feel Earth's gravitational pull and not the Sun's?

10. In the weak field limit the metric tensor g_{ab} differs only slightly from the Minkowski metric η_{ab},

$$g_{ab} = \eta_{ab} + h_{ab} \ ,$$

where h_{ab} is a small correction. Find the Riemann tensor in the lowest order in h_{ab}. Does it satisfy the Bianchi identities?

11. In the weak field limit show that $g_{ab} = \eta_{ab} + h_{ab}$ where $h_{ab \neq 22,33} = 0$, $h_{22} = -h_{33} = f(t-x)$, and f is an arbitrary function, satisfies the (linearized) Einstein equation.

12. In the weak field limit show that $g_{ab} = \eta_{ab} + h_{ab}$ where $h_{ab \neq 23} = 0$, $h_{23} = f(t-x)$, and f is an arbitrary function, satisfies the (linearized) Einstein equation.

13. Two particles at rest are separated spatially by $dx^{\alpha}|_{\alpha=1,2,3} = \{0, dy, 0\}$. A gravitational wave $h_{22} = -h_{33} = f(t-x)$ is passing by. What happens to the proper distance between the particles in this gravitational wave?

14. Tensor F^{ab} is antisymmetric under the exchange of the indices. In special relativity show that

$$F_{a\ \ ,c}^{\ b} F^{c}_{\ b} = -F_{ab,c} F^{bc} \ .$$

15. In a frame with coordinates x^a the invariant interval has the form $ds^2 = \eta_{ab} dx^a dx^b$. Show that after a (smooth and invertible) coordinate transformation $x^a \to \tilde{x}^a$ the interval will take the form $ds^2 = \tilde{g}_{ab} d\tilde{x}^a d\tilde{x}^b$, and also calculate the elements of the tensor $\tilde{g}_{ab}$. Show that for arbitrary vectors A^a and B^b

$$A^a B^b \eta_{ab} = \tilde{A}^a \tilde{B}^b \tilde{g}_{ab} \ .$$

16. Show that if Λ^a_b and $\tilde{\Lambda}^a_b$ are Lorentz transformation matrices than $\Lambda^a_c \tilde{\Lambda}^c_b$ is also a Lorentz transformation matrix.