

Spherically symmetric static solution

Schwarzschild metric

Schwarzschild metric describes the gravitational field outside a non-rotating spherical body like a star, planet, or black hole. It is a static, spherically symmetric solution of the vacuum Einstein equation ($R_{ab} = 0$).

The spherically symmetric static metric is assumed to be in the form

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where A and B are some yet unknown functions of radius r . Calculating the Christoffel symbols¹ and the Ricci tensor² and then integrating³ the vacuum Einstein equations, $R_{ab} = 0$, gives the famous Schwarzschild metric,

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2)$$

The integration constant r_g is determined from the Newtonian limit⁴, $r_g = 2GM/c^2$, where M is the mass of the central body ($r_g = 2M$ in the units $G = c = 1$). It is called gravitational or Schwarzschild radius. The gravitational radius for the Earth is about 9mm, for the Sun – about 3km.

Motion in the Schwarzschild metric

Massive bodies Massive bodies move along geodesics, described by the geodesic equation

$$\frac{d}{ds} (g_{ab}u^b) = \frac{1}{2}g_{bc,a}u^b u^c. \quad (3)$$

For $a = t, \theta, \phi$ the corresponding equations in the Schwarzschild metric (2) are

$$\frac{d}{ds} \left[\left(1 - \frac{2M}{r}\right) \frac{dt}{ds} \right] = 0, \quad (4)$$

$$\frac{d}{ds} \left[r^2 \frac{d\theta}{ds} \right] = r^2 \sin\theta \cos\theta \left(\frac{d\phi}{ds} \right)^2, \quad (5)$$

$$\frac{d}{ds} \left[r^2 \sin^2\theta \frac{d\phi}{ds} \right] = 0. \quad (6)$$

The r -equation can be conveniently obtained by dividing the Schwarzschild metric (2) by ds^2 ,

$$1 = \left(1 - \frac{2M}{r}\right) \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 \left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 \right]. \quad (7)$$

¹ $\Gamma_{rr}^r = \frac{1}{2} \frac{B'}{B}$, $\Gamma_{tr}^t = \frac{1}{2} \frac{A'}{A}$, $\Gamma_{tt}^r = \frac{1}{2} \frac{A'}{B}$, $\Gamma_{\theta r}^\theta = \frac{1}{r}$, $\Gamma_{\theta\theta}^r = -\frac{r}{B}$, $\Gamma_{\phi r}^\phi = \frac{1}{r}$, $\Gamma_{\phi\phi}^r = -\frac{r \sin^2\theta}{B}$, $\Gamma_{\phi\theta}^\phi = \cot\theta$, $\Gamma_{\phi\phi}^\theta = -\sin\theta \cos\theta$.

$$\begin{aligned} R_{tt} &= \frac{A''}{2B} + \frac{A'}{B} \left(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A} \right), \\ R_{\theta\theta} &= 1 - \left(\frac{r}{B} \right)' - \frac{1}{2} \left(\frac{A'}{A} + \frac{B'}{B} \right) \frac{r}{B}, \\ R_{rr} &= -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}. \end{aligned}$$

³ Making a linear combination $BR_{tt} + AR_{rr} = 0$ gives $A'B + AB' = 0 \Rightarrow AB = 1 \Rightarrow \frac{A'}{A} + \frac{B'}{B} = 0$. Then $R_{\theta\theta} = 0$ gives $B = \frac{1}{1 - \frac{r_g}{r}}$, $A = 1 - \frac{r_g}{r}$, where r_g is an integration constant.

⁴ $g_{00} \xrightarrow{r \rightarrow \infty} 1 + 2\phi = 1 - 2\frac{GM}{r}$

Considering equatorial motion, the first three equations can be integrated as

$$\theta = \frac{\pi}{2}, \quad r^2 \frac{d\phi}{ds} = J, \quad \left(1 - \frac{2M}{r}\right) \frac{dt}{ds} = E, \quad (8)$$

where J and E are constants. The fourth equation then becomes

$$1 = \frac{E^2}{1 - \frac{2M}{r}} - \frac{\frac{J^2}{r^4} r'^2}{1 - \frac{2M}{r}} - \frac{J^2}{r^2}, \quad (9)$$

where $r' \equiv \frac{dr}{d\phi}$. Traditionally one makes a variable substitution $r = 1/u$,

$$(1 - 2Mu) = E^2 - J^2 u'^2 - J^2 u^2 (1 - 2Mu), \quad (10)$$

and then differentiates the equation once. Assuming $u' \neq 0$ this gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2. \quad (11)$$

In this form the last term is a relativistic correction to the otherwise non-relativistic equation.

Light rays. The rays of light travel along the null-geodesics where $ds^2 = 0$. Consequently instead of ds one needs to use some parameter $d\lambda$ in the geodesic equations $\frac{Dk^a}{d\lambda} = 0$, where $k^a = \frac{dx^a}{d\lambda}$ and also the unity in the left-hand side of equation (7) has to be substituted with zero. This immediately leads to the equation

$$u'' + u = 3Mu^2, \quad (12)$$

which describes the trajectory of a ray of light in the Schwarzschild metric.

In the absence of the central body, $M = 0$ the space becomes flat, and equation (12) turns into equation for a straight line.

Exercises

1. Consider a non-relativistic equatorial ($\theta = \pi/2$) motion of a planet with mass m around a star with mass M described by a Lagrangian⁵

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mM}{r},$$

Write down the Euler-Lagrange equations,

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q},$$

for $q = r$ and ϕ . Using the first integral $r^2\dot{\phi} = J$ rewrite the r -equation as an equation for the function $u(\phi)$, where $u = 1/r$, and compare with (11).

2. Show that in Newtonian mechanics an equatorial ($\theta = \pi/2$) trajectory of a light ray is described by the equation

$$u'' + u = 0,$$

where $u \doteq \frac{1}{r}$ and $u' \doteq \frac{du}{d\phi}$.

3. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. Answer: $r = \frac{3}{2}(2M)$.
4. Show, that in the Newtonian limit, $g_{00} = 1 + 2\phi/c^2$, the geodesic equation,

$$\frac{du^a}{ds} = -\Gamma_{bc}^a u^b u^c,$$

is consistent with the Newton's equation of motion,

$$\ddot{\vec{r}} = -\vec{\nabla}\phi.$$

⁵where the dot denotes the temporal derivative, $\dot{r} \equiv \frac{dr}{dt}$.