Maxwell equations in gravitational fields

In this section we shell derive the (generally covariant) equations for the electomagnetic field—the Maxwell equations—in curved spaces.

The homogeneous Maxwell equation

The generally covariant form of the homogeneous Maxwell equation can be deduced from its form in Minkowski space,

$$F_{[ab,c]} \doteq F_{ab,c} + F_{bc,a} + F_{ca,b} = 0\,, \tag{1}$$

where

$$F_{ab} \doteq -A_{\{a,b\}} \doteq -A_{a,b} + A_{b,a} \tag{2}$$

is the electromagnetic tensor. Actually, due to the symmetry of Christoffel symbols $\Gamma^a_{bc} = \Gamma^a_{cb}$ (and asymmetry of the electromagnetic tensor) both the electromagnetic tensor and the homogeneous Maxwell equation are already generally covariant and thus preserve their form in curved spaces since

$$-A_{\{a,b\}} = -A_{\{a;b\}}\,, \tag{3}$$

and

$$F_{[ab,c]} = F_{[ab;c]}\,. \tag{4}$$

The inhomogeneous Maxwell equation

For the inhomogeneous Maxwell equation we shall first deduce the generally covariant action of the electromagnetic field, and then derive the equation from the stationary action principle.

In Minkowski space the action of the electromagnetic field is given as an integral over the whole space-time,

$$S = \int d\Omega \left(-\frac{1}{16\pi} F^{ab} F_{ab} - A_a j^a \right)\,. \tag{5}$$

Covariant volume element The infinitesimal volume element,

$$d\Omega \doteq d^4x\,, \tag{6}$$

in this action is not invariant under a general coordinate transformation and has to be substituted with the generally covariant volume element,

$$d\Omega \to \sqrt{-g}\, d\Omega\,, \tag{7}$$

where g is the determinant of the metric tensor g_{ab} ($g < 0$). Indeed, the metric tensor transforms as

$$g_{ab} = \frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} g'_{cd}\,. \tag{8}$$

Taking determinant of both sides gives $g = J'^2 g'$, or

$$\sqrt{-g} = J' \sqrt{-g'}\,, \tag{9}$$

where $J' = \left| \frac{\partial x'^a}{\partial x^b} \right|$ is the Jacobian of the transformation. The 4-volume transforms as

$$d\Omega = \left| \frac{\partial x^a}{\partial x'^b} \right| d\Omega' = \frac{1}{J'} d\Omega'. \tag{10}$$

Apparently the combination $\sqrt{-g}\, d\Omega$ transforms as

$$\sqrt{-g}\, d\Omega = J' \sqrt{-g'} \frac{1}{J'} d\Omega' = \sqrt{-g'}\, d\Omega', \tag{11}$$

and is thus generally invariant.

The generally covariant action of the electromagnetic field is thus given as

$$S = \int \sqrt{-g} d\Omega \left(-\frac{1}{16\pi} F^{ab} F_{ab} - A_a j^a \right) . \tag{12}$$

Variation of the action and the inhomogeneous Maxwell equation The variation of the electromagnetic field, $A_a \to A_a + \delta A_a$, in the second term of the action produces

$$-\delta \int \sqrt{-g} d\Omega A_a j^a = - \int d\Omega \sqrt{-g} j^a \delta A_a . \tag{13}$$

The first term gives

$$-\frac{1}{16\pi} \delta \int \sqrt{-g} d\Omega F^{ab} F_{ab} = +\frac{1}{4\pi} \int d\Omega \sqrt{-g} F^{ab} (\delta A_a)_{,b} = -\frac{1}{4\pi} \int d\Omega \left(\sqrt{-g} F^{ab} \right)_{,b} \delta A_a \tag{14}$$

Combining the two terms gives the variation of the electromagnetic action,

$$\delta S = \int d\Omega \delta A_a \left(-\frac{1}{4\pi} \left(\sqrt{-g} F^{ab} \right)_{,b} - \sqrt{-g} j^a \right) , \tag{15}$$

from which directly follows the sought inhomogeneous Maxwell equation in a curved space,

$$\left(\sqrt{-g} F^{ab} \right)_{,a} = 4\pi \sqrt{-g} j^b . \tag{16}$$

It can also be written in an explicitely covariant form,

$$F^{ab}_{;a} = 4\pi j^b . \tag{17}$$

To prove that, we shall first need the formula for the variation δg of the determinat g of the metric tensor (the Jacobi's formula). Suppose we vary one element, say g_{23}, of the metric tensor. The contribution of this element to the determinant of the metric tensor, g, is given as

$$g_{23} C_{23} = g_{23} g g^{23} , \tag{18}$$

where C_{23} is the cofactor of the element g_{23}, and g^{23} is the element of the inverse metric tensor. Then, apparently,

$$\delta g = g \delta g_{ab} g^{ab} = -g g_{ab} \delta g^{ab} . \tag{19}$$

Now let us consider the covariant divergence of the electromagnetic tensor,

$$F^{ab}_{;a} = F^{ab}_{,a} + \Gamma^a_{da} F^{db} . \tag{20}$$

The contraction Γ^a_{da} of the Christoffel symbol is given as

$$\Gamma^a_{da} = \frac{1}{2} g^{ab} g_{ab,d} = \frac{1}{2g} g_{,d} . \tag{21}$$

The divergence then becomes

$$F^{ab}_{;a} = F^{ab}_{,a} + \frac{1}{2g} g_{,d} F^{db} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} F^{ab} \right)_{,a} , \tag{22}$$

which concludes the proof.

Action in general relativity

In general relativity the gravitational field is the metric tensor g_{ab} as indeed the derivatives of this tensor enter the equations of motion of bodies as apparent gravitational forces.

The equations that determine the gravitational field shall be obtained from the variational principle with the action in the form

$$S = S_M + S_G \,, \tag{23}$$

where S_G is the action for the gravitational field, to be defined later, and S_M is the action for matter[1].

Unlike electromagnetism, where there is an explicit coupling term between electromagnetic fields and charges, in general relativity there is no explicit coupling terms between gravitation and matter.

Indeed, in general relativity gravitation is the geometry of space-time rather than a matter field like the electromagnetic field. The action of the matter has the same form as in special relativity, although written in a generally covariant form.

The matter then couples to gravitation through the metric tensors in the matter action.

The variational principle to determine the gravitational field, created by a given distribution of matter, states that the actual gravitational field, g_{ab}, is that where a small variation of the field, $g_{ab} + \delta g_{ab}$ leads to vanishing variation of the action,

$$\delta S\big|_{g_{ab}\to g_{ab}+\delta g_{ab}} = 0 \,. \tag{24}$$

Action for matter

The action for matter in general relativity has the same form as in special relativity, only rewritten, if needed, in a generally covariant way. Particularly, one has to substitute

$$d\Omega \;\to\; \sqrt{-g}d\Omega \,, \tag{25}$$

$$\partial^a\varphi \;\to\; g^{ab}\partial_b\varphi \,, \tag{26}$$

$$\partial_a A^b \;\to\; D_a A^b \,. \tag{27}$$

For example, the action for the electromagnetic field becomes

$$-\frac{1}{16\pi}\int F^{ab}F_{ab}\sqrt{-g}d\Omega \,, \tag{28}$$

the coupling between the electromagnetic field and currents,

$$-\int A_a j^a \sqrt{-g}d\Omega \,, \tag{29}$$

the action for a hypothetical scalar field ϕ,

$$-\int g^{ab}\partial_a\varphi\partial_b\varphi\sqrt{-g}d\Omega \,. \tag{30}$$

Energy-momentum tensor of matter

The action for matter is typically written in terms of the Lagrangian of the matter, $\mathcal{L}$,

$$S_M = \int \mathcal{L}\sqrt{-g}d\Omega \,. \tag{31}$$

[1] Matter in general relativity is everything else but the gravitational field.

The variation of this action under a variation $g^{ab} \to g^{ab} + \delta g^{ab}$ of the metric tensor can be written in terms of a symmetric tensor T_{ab},

$$\delta S_M \doteq \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega = -\frac{1}{2}\int T^{ab}\delta g_{ab}\sqrt{-g}d\Omega,$$

where[2]

$$\frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab} \doteq \delta(\sqrt{-g}\mathcal{L})\,. \tag{32}$$

The tensor T_{ab} is actually the energy-momentum tensor, since in a flat space it satisfies a conservation law. Indeed, consider an infinitesimal coordinate transformation,

$$x^a \to x'^a = x^a + \epsilon^a\,. \tag{33}$$

The variation of the metric tensor under this transformation can be written as[3]

$$\delta g^{ab} = \epsilon^{a;b} + \epsilon^{b;a}\ ,\ \delta g_{ab} = -\epsilon_{a;b} - \epsilon_{b;a}\,. \tag{38}$$

The variation of the action then takes the form

$$\delta S = \int T_{ab}\epsilon^{a;b}\sqrt{-g}d\Omega\,. \tag{39}$$

Integrating by parts[4],

$$\delta S = -\int T^b_{a;b}\epsilon^a\sqrt{-g}d\Omega\,. \tag{40}$$

The variation of the action under coordinate transformation is zero. Thus the tensor T^a_b satisfies the equation

$$T^{ab}_{;b} = 0\,, \tag{41}$$

which in a flat space turns into the energy-momentum conservation equation $T^{ab}_{,b} = 0$. One can thus assume that the tensor is proportional to the canonical energy-momentum tensor. Direct calculations show that the proportionality factor is equal unity.

Exercises

Remember the notation: $_{;a} \doteq D_a \doteq \frac{D}{dx^a}$ and $_{,a} \doteq \partial_a \doteq \frac{\partial}{dx^a}$.

1. Argue that in Minkowski space the electromagnetic field satisfies the homogeneous Maxwell equation,

   $$F_{[ab,c]} = 0\ ,$$

   where

   $$F_{ab} \doteq -A_{\{a,b\}}\,.$$

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[2]From $g_{ab}g^{bc} = \delta^c_a$ follows $g_{ab}\delta g^{bc} = -\delta g_{ab}g^{bc}$ and therefore $T_{ab}\delta g^{ab} = -T^{ab}\delta g_{ab}$.

[3]Differentiating $x'^a = x^a + \epsilon^a$ gives

$$x'^a_{,b} = \delta^a_b + \epsilon^a_{,b}\,. \tag{34}$$

Substituting this into the transformation rule for the metric tensor,

$$g_{ab}(x) = x'^c_{,a}x'^d_{,b}g_{cd}(x')\,, \tag{35}$$

gives

$$g_{ab}(x) = (\delta^c_a + \epsilon^c_{,a})(\delta^d_b + \epsilon^d_{,b})\left(g'_{cd}(x) + g_{cd,e}\epsilon^e + O((\epsilon)^2)\right) \tag{36}$$

$$= g'_{ab} + g_{ad}\epsilon^d_{,b} + g_{bd}\epsilon^d_{,a} + g_{ab,e}\epsilon^e + O\left((\epsilon)^2\right) = g'_{ab} + \epsilon_{a;b} + \epsilon_{b;a} + O\left((\epsilon)^2\right)\,. \tag{37}$$

[4]using

$$\sqrt{-g}A^c_{;c} = (\sqrt{-g}A^c)_{,c}$$

the total covariant didifferential does not contribute to variation.

2. Argue that in a curved space the electromagneric field satisfies the generally covariant form of this equation,
$$F_{[ab;c]} = 0\ ,$$
where
$$F_{ab} \doteq -A_{\{a;b\}}\ .$$

3. Derive the second Maxwell equation in a curved space,
$$\left(\sqrt{-g}F^{ab}\right)_{,a} = 4\pi\sqrt{-g}j^b\ ,$$
from the action
$$S = \int\left(-\frac{1}{16\pi}F^{ab}F_{ab} - A_a j^a\right)\sqrt{-g}d\Omega\,.$$
Show that the equation can also be written as
$$F^{ab}_{;a} = 4\pi j^b\,.$$
Hints:
   (a) show that $\Gamma^a_{ba} = \frac{1}{2g}g_{,b} = (\ln\sqrt{-g})_{,b}$
   (b) show that $F^{ab}_{;a} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ab})_{,a}$

   Show that from this equation it follows, that
$$\left(\sqrt{-g}j^a\right)_{,a} = 0 = j^a_{;a}\,.$$

4. The Lagrangian density for the electromagnetic field is
$$\mathcal{L} = -\frac{1}{16\pi}F_{ab}F^{ab}.$$
Calculate the corresponding energy-momentum tensor using the usual definition,
$$\delta\left(\sqrt{-g}\mathcal{L}\right)_{g^{ab}\to g^{ab}+\delta g^{ab}} = \frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab}\,.$$
Answer:
$$T_{ab} = \frac{1}{16\pi}\left(F_{cd}F^{cd}g_{ab} - 4F_{ac}F_b{}^c\right)\,.$$

5. In the Minkowski space consider a scalar field φ with action
$$S = \int d\Omega\left(\frac{1}{2}\varphi^{,a}\varphi_{,a} - \frac{1}{2}m^2\varphi^2\right)$$
and calculate its "translation-invariance" energy-momentum tensor,
$$T^a_b = \frac{\partial\mathcal{L}}{\partial\varphi_{,a}}\varphi_{,b} - \mathcal{L}\delta^a_b\,.$$
Rewrite the action in a generally covariant form and calculate its "metric" energy-momentum tensor,
$$\delta\left(\sqrt{-g}\mathcal{L}\right)_{g^{ab}\to g^{ab}+\delta g^{ab}} = \frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab}$$
Discuss the results.

6. Show that for infinitesimally small coordinate transformation

$$x^a \to x^a + \epsilon^a$$

the variation of the metric tensor is given as

$$\delta g^{ab} = \epsilon^{[a;b]} \ , \ , \delta g_{ab} = -\epsilon_{[a;b]} \ .$$

7. Derive the energy-momentum tensor of a particle with mass m.
8. Argue that in the flat space the equation $j^a_{,a}$ represents a conservation law.