

## The principle of stationary action

*Action* is an attribute of the dynamics of a physical system. It is a functional which takes the trajectory of the system (also called *path* or *history*) as its argument and returns a covariant real scalar as the result<sup>1</sup>. Generally, the action takes different values for different paths.

Classical mechanics postulates that

*The path actually followed by a physical system is that for which the action is stationary. In other words for which the variation of the action vanishes,*

$$\delta S = 0. \quad (2)$$

Vanishing variation means the action has an extremum on the actual trajectory. The postulate is called the *principle of stationary action* or the *variational principle*.

The classical equations of motion can be derived from the variational principle.

## Motion of a charged body in gravitational and electromagnetic fields

In special relativity the action of a body with mass  $m$  and charge  $e$ , moving in a given electromagnetic field  $A^a$ , is given (in the units  $c = 1$ ) as

$$S = -m \int ds - e \int dx^a A_a, \quad (3)$$

where the integrals are taken along the trajectory of the body. In this form the action is generally covariant and can be directly used in general relativity. One only has to remember that the metric tensor is not constant throughout the curved space.

We have to calculate the variation of this action under a small variation of the trajectory of the body,

$$x^a \rightarrow x^a + \delta x^a. \quad (4)$$

The variation of the first term in (3) has already been calculated in the section about geodesics,

$$\delta \left( -m \int ds \right) = \int ds \delta x^a m \left( \frac{du_a}{ds} - \frac{1}{2} g_{bc,a} u^b u^c \right). \quad (5)$$

The variation of the term  $dx^a A_a$  is given as

$$\delta(dx^a A_a) = \delta dx^a A_a + u^a ds A_{a,b} \delta x^b = \delta dx^a A_a + u^b ds A_{b,a} \delta x^a. \quad (6)$$

As usual the term  $\delta dx^a A_a$  is integrated by parts,

$$\delta dx^a A_a = d(\delta x^a A_a) - \delta x^a dA_a = d(\delta x^a A_a) - \delta x^a A_{a,b} u^b ds, \quad (7)$$

which leads to

$$\delta(dx^a A_a) = d(\delta x^a A_a) + (-A_{a,b} + A_{b,a}) u^b ds \delta x^a. \quad (8)$$

The expression in parentheses is called the *electromagnetic tensor*,

$$F_{ab} \doteq -A_{a,b} + A_{b,a}. \quad (9)$$

It is apparently antisymmetric,  $F_{ab} = -F_{ba}$ . Although it contains non-covariant derivatives it is actually a tensor since in a torsion free space it can be also written through covariant derivatives,

$$F_{ab} = -A_{a;b} + A_{b;a}. \quad (10)$$

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<sup>1</sup>If the action is represented as an integral over time, taken along the path of the system between the initial time and the final time of the development of the system,

$$S = \int \mathcal{L} dt, \quad (1)$$

the integrand  $\mathcal{L}$  is called the *Lagrangian*.

The full differentials in (8) as usual does not contribute to the variation since  $\delta x^a$  is zero at the end-points of the trajectory. The variation of the action (3 then becomes

$$\delta S = \int ds \delta x^a \left( m \frac{du_a}{ds} - m \frac{1}{2} g_{bc,a} u^b u^c - e F_{ab} u^b \right). \quad (11)$$

Since the variation  $\delta x^a$  is arbitrary,  $\delta S = 0$  means the expression in parentheses has to vanish identically on physical trajectories, giving the the equation of motion of a charged body in both gravitational and electro-magnetic fields (the generalization of the *Lorentz force* equation),

$$m \frac{du_a}{ds} - m \frac{1}{2} g_{bc,a} u^b u^c = e F_{ab} u^b. \quad (12)$$

It can also be written as

$$m \frac{du_a}{ds} - m \Gamma_{bca} u^b u^c = e F_{ab} u^b, \quad (13)$$

or as

$$m \frac{Du_a}{ds} = e F_{ab} u^b. \quad (14)$$

**Maxwell equations in Minkowski space** In classical electrodynamics the action for the electromagnetic field  $A^a$  with given sources  $j^a$  is written as an integral over the 4-volume,

$$S = -\frac{1}{8\pi} \int d^4x A_{a,b} A^{a,b} - \int d^4x A_a j^a. \quad (15)$$

Canonical calculation of the variation of the action under infinitesimal variation of the field  $A^a \rightarrow A^a + \delta A^a$  gives the second Maxwell equation,

$$A_{,b}^{a,b} = 4\pi j^a. \quad (16)$$

### Exercises

1. Argue that the electromagnetic tensor is actually a tensor by proving that

$$F_{ab} \doteq -A_{a,b} + A_{b,a} = -A_{a;b} + A_{b;a}. \quad (17)$$

2. Argue that

$$\frac{du_a}{ds} - \frac{1}{2} g_{bc,a} u^b u^c = \frac{du_a}{ds} - \Gamma_{bca} u^b u^c. \quad (18)$$

3. Derive the Lorentz force equation from the action  $S = -m \int ds - e \int dx^a A_a$  in the Minkowski space of special relativity. Rewrite this equation in 3-notation where

$$A^a = \{\phi, \vec{A}\}, \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{H} = \text{curl } \vec{A} \equiv \vec{\nabla} \times \vec{A}.$$

4. In Minkowski space of special relativity from the action

$$S = -\frac{1}{8\pi} \int d^4x A_{a,b} A^{a,b} - \int d^4x A^a j_a$$

derive<sup>2</sup> the Maxwell equations with sources,

$$A_{,a}^{b,a} = 4\pi j^b.$$

<sup>2</sup>For the fields the usual “integration by parts” is done using the Gauss theorem in Minkowski space,

$$\int_{\Omega} d^4x \frac{\partial B^a}{\partial x^a} = \oint_{\partial\Omega} B^a dS_a,$$

where  $dS_a$  is an infinitesimal element of the hyper-surface  $\partial\Omega$  of the 4-volume  $\Omega$ . In Euclidean 3D space it has a more familiar form,

$$\int_V dV (\vec{\nabla} \cdot \vec{B}) = \int_{\partial V} dS (\vec{B} \cdot \vec{n}).$$

Show that with the Lorenz condition,

$$A_{,a}^a = 0,$$

it is equivalent to

$$F_{,a}^{ab} = 4\pi j^b.$$

Write down the latter in 3-notation.