

Friedman (FLRW) universe

The *Friedman universe* is a cosmological model where the universe is assumed to be homogeneous, isotropic, and filled with *ideal fluid*. The assumption of homogeneity and isotropicity of the universe is often referred to as the *cosmological principle*. Empirically, it seems to be justified on scales larger than 100 Mpc. The isotropic and homogeneous model is sometimes called the *Standard Model* of present-day cosmology. It is most often referred to as Friedman-Lemaitre-Robertson-Walker model (or FLRW model, for short).

The *Friedman equation* is the Einstein equation applied to the Friedman universe as a whole. It describes the temporal evolution of a Friedman universe.

Spaces with constant curvature

A homogeneous and isotropic universe is a *space with constant curvature*.

Two-dimensional spaces of constant curvature are three-dimensional spheres (positive curvature), pseudo-spheres (negative curvature), and planes (zero curvature).

On a sphere the length element in the ordinary spherical coordinates is given as

$$dl^2 = a^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (1)$$

where a is the radius of the sphere.

Let us introduce the polar coordinates $\{r, \phi\}$ on the sphere, with r measuring the distance to the north pole. The length of a circle around north pole, $\theta = \text{const}$, is equal $2\pi a \sin \theta$. Therefore if we want the circumference of the circle to be equal $2\pi r$, we need to define $r = a \sin \theta$.

The length element in the polar coordinates becomes

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 d\phi^2 . \quad (2)$$

On a pseudo-sphere, where the curvature is negative, the length element is correspondingly

$$dl^2 = \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2 d\phi^2 . \quad (3)$$

In angular coordinates $r = a \sinh \theta$ the latter becomes

$$dl^2 = a^2 (d\theta^2 + \sinh^2 \theta d\phi^2) . \quad (4)$$

On a plane, where the curvature is zero, the length element is

$$dl^2 = dr^2 + r^2 d\phi^2 . \quad (5)$$

Analogously, a three-dimensional space with constant curvature can have one of the following three possible geometries:

- *flat* (zero curvature),

$$dl^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ; \quad (6)$$

- *closed* (positive curvature),

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) = a^2 (d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad (7)$$

where $r = a \sin \chi$, $\chi \in [0, \pi]$;

- and *open* (negative curvature),

$$dl^2 = \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) = a^2 (d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad (8)$$

where $r = a \sinh \chi$, $\chi \in [0, \infty[$.

Friedman equation

*Friedman metric*¹ is a (generic) synchronous metric in a homogeneous and isotropic universe,

$$ds^2 = dt^2 - a(t)^2 dl^2, \quad (9)$$

where dl^2 is the line element in a three-dimensional space of constant curvature, and $a(t)$ is the time-dependent *scale factor* of the universe.

Closed universe In a closed Friedman universe the metric is

$$ds^2 = a^2 (d\eta^2 - d\chi^2 - \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)), \quad (10)$$

where $r = a \sin \chi$, and η is the scaled time coordinate,

$$dt = a d\eta. \quad (11)$$

Now the following Maxima script

```
derivabbrev:true; /* a better notation for derivatives */
load(ctensor); /* load the package to deal with tensors */
ct_coords:[n,k,o,p]; /* our coordiantes: eta chi theta phi */
depends([a],[n]); /* the scale, a, depends on eta */
lg:ident(4); /* set up a 4x4 identity matrix */
lg[1,1]:a^2; /* g_{\eta\eta} = a^2 */
lg[2,2]:-a^2; /* g_{\chi\chi} = -a^2 */
lg[3,3]:-a^2*sin(k)^2; /* g_{\chi\chi} */
lg[4,4]:-a^2*sin(k)^2*sin(o)^2; /* g_{\phi\phi} */
cmetric(); /* compute the prerequisites for further calculations */
christof(mcs); /* print out the Christoffel symbols */
uricci(true); /* print out the elements of the Ricci tensor */
scurvature(); /* Ricci scalar curvature */
```

calculates the components of the Ricci tensor² and the Ricci scalar for this metric,

$$R_\eta^\eta = \frac{3}{a^4}(a'^2 - aa''), \quad R_\chi^\chi = R_\theta^\theta = R_\phi^\phi = -\frac{1}{a^4}(2a^2 + a'^2 + aa''), \quad R = -\frac{6}{a^3}(a + a''), \quad (12)$$

where prime denotes the η -derivative.

The energy-momentum tensor for the perfect fluid is

$$T_{ab} = (\epsilon + p)u_a u_b - p g_{ab}, \quad (13)$$

where ϵ is the energy density and p is the pressure. In synchronous Friedman coordinates the matter is at rest and the 4-velocity is $u^b = \{\frac{1}{a}, 0, 0, 0\}$.

Correspondingly, the Einstein equation

$$R_b^a - \frac{1}{2}R\delta_b^a = \kappa T_b^a \quad (14)$$

then has the $\frac{\eta}{\eta}$ component

$$\frac{3}{a^4}(a'^2 + a'^2) = \kappa\epsilon, \quad (15)$$

¹also referred to as Friedman-Lemaitre-Robertson-Walker in different combinations.

²The script also calculates the Christoffel symbols, $\Gamma_{\eta\eta}^\eta = \Gamma_{\eta\chi}^\chi = \Gamma_{\eta\theta}^\theta = \Gamma_{\eta\phi}^\phi = \Gamma_{\chi\chi}^\eta = \frac{a'}{a}$, $\Gamma_{\chi\theta}^\theta = \Gamma_{\chi\phi}^\phi = \cot \chi$, $\Gamma_{\theta\theta}^\eta = \frac{a'}{a} \sin^2 \chi$, $\Gamma_{\theta\theta}^\chi = -\cos \chi \sin \chi$, $\Gamma_{\theta\phi}^\phi = \cot \theta$, $\Gamma_{\phi\phi}^\eta = \frac{a'}{a} \sin^2 \chi \sin^2 \theta$, $\Gamma_{\phi\phi}^\chi = -\cos \chi \sin \chi \sin^2 \theta$, $\Gamma_{\phi\phi}^\theta = -\cos \theta \sin \theta$.

and the three identical spatial components,

$$\frac{1}{a^4}(a^2 + 2aa'' - a'^2) = -\kappa p, \quad (16)$$

called the *Friedman equations* for a closed universe,

If the relation between ϵ and p , called the *equation of state* of the matter, is known, the energy density ϵ can be determined as function of a from the energy conservation equation. The latter must have the form

$$dE = -pdV, \quad (17)$$

where V is a volume element in the Friedman universe, and $E = \epsilon V$ is the energy content of this volume. Since the volume is proportional to a^3 , and both ϵ and a in a Friedman universe can only depend on time, equation (17) can be rewritten as

$$(\epsilon a^3)' + p(a^3)' = 0. \quad (18)$$

It is easy to show, that equation (18) actually follows from the Friedman equations (15) and (16).

The energy conservation equation (17) can also be written as

$$\frac{3da}{a} = -\frac{d\epsilon}{\epsilon + p}. \quad (19)$$

When the dependence $\epsilon(a)$ is found by integration of the energy conservation equation (19), the solution to the Friedman equation can be obtained as the integral

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 - 1}}. \quad (20)$$

Open universe In an open Friedman universe the metric is

$$ds^2 = a^2 (d\eta^2 - d\chi^2 - \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)), \quad (21)$$

where $r = a \sinh \chi$, and $ad\eta = dt$. This metric can be obtained from the closed universe metric (10) by a formal substitution

$$\{a, \eta, \chi\} \rightarrow \{ia, i\eta, i\chi\}. \quad (22)$$

Therefore the Friedman equation for an open universe can be readily obtained from (15) by the substitution (22),

$$\frac{3}{a^4}(a'^2 - a^2) = \kappa\epsilon, \quad (23)$$

with the integral solution,

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 + 1}}. \quad (24)$$

Exercises

1. Calculate the (diagonal components of the) Ricci tensor in Friedman coordinates for a closed universe.
2. Consider a flat (Euclidean) isotropic universe with the metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2).$$

and investigate its temporal development for matter and radiation dominated universes.

- (a) Calculate the Christoffel symbols,

$$\Gamma_{tx}^x = \Gamma_{ty}^y = \Gamma_{tz}^z = \frac{\dot{a}}{a}, \quad \Gamma_{xx}^t = \Gamma_{yy}^t = \Gamma_{zz}^t = a\dot{a}.$$

- (b) Calculate the Ricci tensor and the Ricci scalar,

$$R_t^t = -3\frac{\ddot{a}}{a}, \quad R_x^x = R_y^y = R_z^z = -\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2}.$$

- (c) Write down the t_t component of the Einstein equation with perfect fluid,

$$3\frac{\dot{a}^2}{a^2} = \kappa\epsilon.$$

- (d) Write down the energy conservation equation,

$$\frac{dV}{V} = -\frac{d\epsilon}{(\epsilon + p)} \Rightarrow 3\ln(a) = -\int \frac{d\epsilon}{(\epsilon + p)}.$$

- (e) Integrate the equations for a matter dominated universe ($p = 0$, $\epsilon = \mu$),

$$\mu a^3 = \text{const}, \quad a = \propto t^{2/3}.$$

- (f) Integrate the equations for a radiation dominated universe ($p = \frac{\epsilon}{3}$),

$$\epsilon a^4 = \text{const}, \quad a = \propto t^{1/2}$$

3. Calculate the volumes of the closed and open universes.