1-i A vector has components $A^a = \{0, -2, 3, 5\}$ in the frame K. What are the components of the vector in a frame which moves with the velocity $0.8c$ in the positive z direction relative to K.

1-ii Two events in special relativity are separated by a space-like interval, $ds^2 < 0$. Is there an inertial frame where the two events are simultaneous?

1-iii In special relativity: argue that a free electron can neither absorb nor emit a photon.

1-iv Consider a 3-dimensional Euclidean space with Cartesian coordinates. Is there a difference between vectors with index up and index down?

1-v Consider a vector A^a. Is the four-component object $\left\{\frac{1}{A^0}, \frac{1}{A^1}, \frac{1}{A^2}, \frac{1}{A^3}\right\}$ a vector?

2-i In Schwarzschild coordinates $\{t, r\}$ the half-life of an elementary particle measured at rest at large distance from the center is Δt. What is its half-life at rest at a distance r from the center? What is the proper half-life?

2-ii Find out how the expression $A^a{}_{,b}$ transforms under a change of coordinates. Does it obey the tensor transformation law?

2-iii Show that the Christoffel symbol Γ^a_{bc} is symmetric under exchange of the lower indices.

2-iv Consider a scalar function of coordinates $\phi(x)$. Find out whether the objects $\frac{\partial\phi}{\partial x^a}$ and $\frac{\partial^2\phi}{\partial x^a \partial x^b}$ are tensors in special and general relativity.

2-v Prove that in a locally inertial frame

$$R_{abcd,e} = \frac{1}{2}(g_{ad,bce} - g_{ac,bde} + g_{bc,ade} - g_{bd,ace}) \,.$$

2-vi Starting with Minkowski metric $ds^2 = \eta_{ab}dx^a dx^b$, show that the coordinate transformation $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arccos(z/r)$, $\varphi = \arctan(y/x)$ leads to metric $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$.

3-i Assume that the components of the metric tensor g_{ab} do not depend on the coordinate x^1. Show that for a free moving body the component u_1 of the body's four-velocity is then conserved.

3-ii Show that Maxwell equations $F^{ba}_{\ \ ;b} = 4\pi j^a$ it follows that $j^a_{\ ;a} = 0$. Hint: prove first that $\sqrt{-g}A^c{}_{;c} = (\sqrt{-g}A^c)_{,c}$.

3-iii Use the definition of the invariant volume element (called covariant in lecture notes) to calculate the proper area of a sphere.

3-iv Show that for an equatorial orbit in the Schwarzschild metric the quantity u_φ^2 is conserved (where u^φ is the $\varphi-$component of the four-velocity, $u^a = dx^a/ds$).

4-i In the weak field limit show that $g_{ab} = \eta_{ab} + h_{ab}$ where $h_{yz} = A \sin \omega(t-x)$, $h_{tt} = 2f(t-x)$, $h_{tx} = -f(t-x)$, all other $h_{ab} = 0$, f is an arbitrary function, satisfies the linearized Einstein equation in vacuum. Hint: see the gravitational waves chapter.

4-ii Let us define the generalized force $\mathcal{F}_a$ acting on a particle with mass m through the equation of motion $mDu_a/ds = \mathcal{F}_a$. Recall that from $u^a u_a = 1$ it follows that $Du_a u^a = 0$ and therefore $\mathcal{F}_a u^a = 0$. Show that the electromagnetic force satisfies this condition. Show that a force in the form $\mathcal{F}_a = -\partial V/\partial x^a$, where V is a scalar function of coordinates, generally does not satisfy this condition. What would a discovery of such force mean?

4-iii Consider a nearly circular orbit of a planet around a star in Newtonian mechanics and in General Relativity. Derive the equation for a small radial perturbation of the orbit and find its angular period. Relate to the post-Newtonian perihelion precession.

4-iv Consider a flat radiation-dominated Friedman universe. Show that at early times it expands as $a \propto \sqrt{t}$.