1-1 A vector has zero components in one frame. Are the components zero in *all* other frames? Give the answers for special and general relativity.

1-2 Two events in special relativity are separated by a time-like interval, $ds^2 > 0$. Is there an inertial frame where the two events are simultaneous?

1-3 A vector has components $A^a = \{0, -2, 3, 5\}$ in a given frame K. What are the components of the vector in a frame which moves with the velocity $0.6c$ in the positive y direction relative to K.

1-4 Suppose that the relative velocity v between two inertial frames is close to the speed of light, $v = c - \varepsilon$, where $0 < \varepsilon \ll c$. Derive the formulas for the Lorentz contraction and time-dilation in the lowest order in ε. What are the relative errors of these formulas when $\varepsilon = 0.1c$?

1-5 Argue that the space of special relativity is a metric space with the metric tensor η_{ab} (a diagonal tensor where the components of the main diagonal are equal $(1, -1, -1, -1)$).

2-1 In special relativity: consider the 4-velocity $u^a = dx^a/ds$ and the 3-velocity $\vec{v} = d\vec{r}/dt$ of a moving body with coordinates $x^a = \{t, \vec{r}\}$. Express u^a in terms of $\vec{v}$.

2-2 Calculate the components of g^{ab} for a sphere in polar coordinates.

2-3 In a two-dimensional Euclidean space with polar coordinates the components of a vector field are given as $A^r = 1$, $A^\theta = 0$. Compute $A^a{}_{;b;c}$.

2-4 Consider a tensor which is symmetric, $S_{ab} = S_{ba}$, in a given frame. Is it symmetric in other frames? What can one say about the symmetry of the tensor S^{ab} (with indexes up)?

2-5 Prove that $\Gamma^a{}_{ab} = \frac{1}{2}(\ln|g|)_{,b}$.

3-1 Calculate the components of the Riemann tensor for a 2-dimensional Euclidean space with polar coordinates.

3-2 Assume that a geodesic is a curve with extremal measure, that is, a curve along which

$$\delta \int ds = 0\,.$$

Derive the geodesic equation. Prove that the covariant differential of the velocity of a particle moving along the geodesic is zero, $Du^a = 0$.

3-3 Calculate—using the metric—the length of a circle of constant coordinate θ on a sphere of radius r.

3-4 The metric in a space is given as

$$ds^2 = (1 + 2\phi)dt^2 - (1 - 2\phi)(dx^2 + dy^2 + dz^2)\ ,$$

where $|\phi| \ll 1$ everywhere. To first order in ϕ compute the Christoffel symbols. Assume that ϕ is a function of (t, x, y, z).

4-1 In Schwarzschild coordinates calculate the period of a circular orbit with radius b of a planet rotating around a star of mass M. Calculate also the proper period of the orbit. Compare with Newtonian result. Hint: recall the Kepler's law exercise.

4-2 The metric in a space is given as

$$ds^2 = (1 + 2\phi)dt^2 - (1 - 2\phi)(dx^2 + dy^2 + dz^2) \ ,$$

where $|\phi| \ll 1$ everywhere. At a given point (t_0, x_0, y_0, z_0) find a coordinate transformation to a locally inertial frame, to first order in ϕ. At what rate does this frame accelerates with respect to the original coordinates, again to the first order in ϕ?

4-3 A rocket fell through the gravitational radius of a black hole and tries to escape back. Show that it will reach the center within the proper time $s \leq \pi M$ no matter how powerful the rocket engines are. Hints: argue that for a body under the Schwarzschild radius always $dr < 0$; from the condition $g_{ab}u^a u^b = 1$ prove the following inequality under the Schwarzschild radius,

$$\left(\frac{2M}{r} - 1\right)^{-1} \left(\frac{dr}{ds}\right)^2 > 1 \, ,$$

from which you can obtain the limit on the proper velocity dr/ds; Maxima claims that

$$\int \frac{dr}{\sqrt{\frac{1}{r} - 1}} = -r\sqrt{\frac{1}{r} - 1} - \arctan\sqrt{\frac{1}{r} - 1} \, .$$

4-4 In Newtonian mechanics consider a planet rotating around a star which is slightly non-spherical, such that the classical Newtonian gravitational potential is

$$\phi(r) = -\frac{M}{r} - \frac{AM}{r^3} \ , \qquad (1)$$

where the small parameter A describes the non-sphericity of the star. In Newtonian mechanics calculate the precession of the perihelion of the orbit of the planet. For simplicity you can assume that the orbit is nearly circular.