

- 1-a Let  $\Lambda_b^a$  be the matrix of the Lorentz transformation from a frame  $K$  to a frame  $K'$ . What is the matrix of the Lorentz transformation from  $K'$  to  $K$ ?
- 1-b Two vectors have equal components in one frame. Do they have equal components in other frames? Give the answers for special and general relativity.
- 1-c Two events in special relativity are separated by a time-like interval,  $ds^2 > 0$ . Is there an inertial frame where the two events occur at the same spatial point?
- 1-d Consider a 3-dimensional Euclidean space with polar coordinates. Is there a difference between vectors with index up and vectors with index down?
- 1-e Consider a tensor which is antisymmetric,  $F_{ab} = -F_{ba}$ , in a given frame. Is it antisymmetric in other frames? What can one say about the symmetry of the tensor  $F^{ab}$  (with indexes up)?
- 2-a In some inertial frame the motion of a particle is described by the equations

$$x(t) = at + b \sin(\omega t), \quad y(t) = b \cos(\omega t), \quad z(t) = 0, \quad |b\omega| < 1.$$

Compute the components of particle's four-velocity and four-acceleration.

- 2-b Calculate the Riemann tensor for a cylinder. You are free to choose the system of coordinates.
- 2-c Consider a tensor  $X_{bc}^a$ . Is the quantity  $Y_c = X_{ac}^a$  a tensor?
- 2-d In a two-dimensional Euclidean space with polar coordinates consider a tensor  $A^{ab}$  with the following components:  $A^{rr} = r^2$ ,  $A^{r\theta} = r \sin \theta$ ,  $A^{\theta r} = r \cos \theta$ ,  $A^{\theta\theta} = \tan \theta$ . Compute the covariant derivatives  $A_{;c}^{ab}$  (in polar coordinates).
- 2-e Differentiate a determinant of a  $2 \times 2$  matrix and show that it satisfies the equation  $g_{,c} = gg^{ab}g_{ab,c}$ .
- 2-f Consider a space with coordinates  $(u, v, w, p)$  and metric  $ds^2 = 2dudv - dvdw - dpdp$ . Show that this is Minkowski space.
- 3-a A traveller starts from rest at the Earth and moves along a line with constant acceleration  $a$  with respect to the momentarily co-moving inertial frame (also called the instantaneous rest frame). Argue that light signals sent from the Earth after time  $t = c/a$  will never reach the receding traveller.
- 3-b The non-relativistic Lagrangian of a free particle with mass  $m$  is given as  $L = \frac{1}{2}m\vec{v}^2$ , where  $\vec{v}$  is the particle's velocity. What is the corresponding relativistic Lagrangian? What is the connection between the non-relativistic and the relativistic Lagrangians?

3-c The metric in a space is given as

$$ds^2 = (1 + 2\phi)dt^2 - (1 - 2\phi)(dx^2 + dy^2 + dz^2),$$

where  $|\phi| \ll 1$  everywhere. To first order in  $\phi$  compute  $g^{ab}$ .

3-d Prove that in a locally inertial frame

$$R_{abcd,e} = \frac{1}{2}(g_{ad,bce} - g_{ac,bde} + g_{bc,ade} - g_{bd,ace}).$$

4-a Consider a massless scalar field  $\Phi$  with the energy-momentum tensor

$$T_{ab} = \frac{1}{4\pi} \left( \Phi_{,a}\Phi_{,b} - \frac{1}{2}g_{ab}\Phi_{,c}\Phi^{,c} \right).$$

From the equation  $T_{;b}^{ab} = 0$  derive the equations of motion for the field  $\Phi$ .

4-b The space is everywhere isotropic and empty (contains no matter at all). Show that it is Minkowski space. Hint: consider the Friedman equations.

4-c In the weak field limit show that the metric tensor  $g_{ab} = \eta_{ab} + h_{ab}$  where  $h_{ab \neq yz, zy} = 0$ ,  $h_{yz} = f(t - x)$ , and  $f$  is an arbitrary function, satisfies the linearized Einstein equation in vacuum. Hint: see the gravitational waves chapter.

4-d Show that the equatorial orbit in the Schwarzschild metric is stable. Hint: consider an equatorial orbit with a small perturbation  $\theta = \pi/2 + \delta\theta$ ; derive the lowest order equation for the perturbation  $\delta\theta$ ; show that the perturbation remains small.