Spherically symmetric static solution

Schwarzschild metric

Schwarzschild metric describes the gravitational field outside a non-rotating spherical body like a star, planet, or black hole. It is a static, spherically symmetric solution of the vacuum Einstein equation ($R_{ab} = 0$).

The spherically symmetric static metric is assumed to be in the form

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1}$$

where A and B are some yet unknown functions of radius r. Calculating the Christoffel symbols[1] and the Ricci tensor[2] and then integrating[3] the vacuum Einstein equations, $R_{ab} = 0$, gives the famous Schwarzschild metric,

$$ds^2 = \left(1 - \frac{R}{r}\right)dt^2 - \left(1 - \frac{R}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2}$$

The integration constant R is determined from the Newtonian limit[4], $R = 2GM/c^2$, where M is the mass of the central body ($R = 2M$ in the units $G = c = 1$). It is called gravitational or Schwarzschild radius. The gravitational radius for the Earth is about 9mm, for the Sun – about 3km.

Motion in the Schwarzschild metric

Massive bodies. Massive bodies move along geodesics, described by the geodesic equation

$$\frac{d}{ds}(g_{ab}u^b) = \frac{1}{2}g_{bc,a}u^b u^c\,. \tag{3}$$

For $a = t, \theta, \phi$ the corresponding equations in the Schwarzschild metric (2) are

$$\frac{d}{ds}\left[\left(1 - \frac{2M}{r}\right)\frac{dt}{ds}\right] = 0\ ,\ \frac{d}{ds}\left[r^2\frac{d\theta}{ds}\right] = r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2\ ,\ \frac{d}{ds}\left[r^2\sin^2\theta\frac{d\phi}{ds}\right] = 0\ . \tag{4}$$

The r-equation can be conveniently obtained by dividing the Schwarzschild metric (2) by ds^2,

$$1 = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right]. \tag{5}$$

Considering equatorial motion, the first three equations can be integrated as

$$\theta = \frac{\pi}{2}\ ,\ \ r^2\frac{d\phi}{ds} = J\ ,\ \ \left(1 - \frac{2M}{r}\right)\frac{dt}{ds} = E\ , \tag{6}$$

where J and E are constants. The fourth equation then becomes

$$1 = \frac{E^2}{1 - \frac{2M}{r}} - \frac{\frac{J^2}{r^4}r'^2}{1 - \frac{2M}{r}} - \frac{J^2}{r^2}\ , \tag{7}$$

[1] $\Gamma^r_{rr} = \frac{1}{2}\frac{B'}{B}\,,\ \Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A}\,,\ \Gamma^r_{tt} = \frac{1}{2}\frac{A'}{B}\,,\ \Gamma^\theta_{\theta r} = \frac{1}{r}\,,\ \Gamma^r_{\theta\theta} = -\frac{r}{B}\,,\ \Gamma^\phi_{\phi r} = \frac{1}{r}\,,\ \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B}\,,\ \Gamma^\phi_{\phi\theta} = \cot\theta\,,\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,.$

[2] $R_{tt} = \frac{A''}{2B} + \frac{A'}{B}(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A})\,,\ R_{\theta\theta} = 1 - (\frac{r}{B})' - \frac{1}{2}(\frac{A'}{A} + \frac{B'}{B})\frac{r}{B}\,,\ R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}\,.$

[3] Making a linear combination $BR_{tt} + AR_{rr} = 0$ gives $A'B + AB' = 0 \ \Rightarrow\ \frac{A'}{A} + \frac{B'}{B} = 0 \ \Rightarrow\ AB = 1$. Then $R_{\theta\theta} = 0$ gives $B = \frac{1}{1-\frac{R}{r}}$, $A = 1 - \frac{R}{r}$, where R is an integration constant.

[4] $g_{00} \stackrel{r\to\infty}{\longrightarrow} 1 + 2\phi = 1 - 2\frac{GM}{r}$

where $r' \equiv \frac{dr}{d\phi}$. Traditionally one makes a variable substitution $r = 1/u$,

$$(1-2Mu) = E^2 - J^2u'^2 - J^2u^2(1-2Mu), \tag{8}$$

and then differentiates the equation once. Assuming $u' \neq 0$ this gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2 \,. \tag{9}$$

In this form the last term is a relativistic correction to the otherwise non-relativistic equation.

Light rays. The rays of light travel along the null-geodesics where $ds^2 = 0$. Consequently instead of ds one needs to use some parameter $d\lambda$ in the geodesic equations $\frac{Dk^a}{d\lambda} = 0$, where $k^a = \frac{dx^a}{d\lambda}$ and also the unity in the left-hand side of equation (5) has to be substituted with zero. This immediately leads to the equation

$$u'' + u = 3Mu^2 \,, \tag{10}$$

which describes the trajectory of a ray of light in the Schwarzschild metric.

In the absence of the central body, $M = 0$ the space becomes flat, and equation (10) turns into equation for a straight line.

Exercises

1. (**Obligatory**) Consider a non-relativistic equatorial ($\theta = \pi/2$) motion of a planet with mass m around a star with mass M described by a Lagrangian[5]

   $$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mM}{r} \,,$$

   Write down the Euler-Lagrange equations,

   $$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \,,$$

   for $q = r$ and ϕ. Using the first integral $r^2\dot{\phi} = J$ rewrite the r-equation as an equation for the function $u(\phi)$, where $u = 1/r$, and compare with (9).

2. Show that in Newtonian mechanics an equatorial ($\theta = \pi/2$) trajectory of a light ray is described by the equation

   $$u'' + u = 0 \,,$$

   where $u \doteq \frac{1}{r}$ and $u' \doteq \frac{du}{d\phi}$.

3. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. Answer: $r = \frac{3}{2}(2M)$.

4. (**Obligatory**) Show, that in the Newtonian limit, $g_{00} = 1 + 2\phi/c^2$, the geodesic equation,

   $$\frac{du^a}{ds} = -\Gamma^a_{bc}u^bu^c \,,$$

   is consistent with the Newton's equation of motion,

   $$\ddot{\vec{r}} = -\vec{\nabla}\phi \,.$$

[5] where the dot denotes the temporal derivative, $\dot{r} \equiv \frac{dr}{dt}$.