Newtonian limit

Newtonian gravitation

The Newton's law of gravitation states that two particles with masses m and M located at a relative distance r attract each other with the force

$$F = G\frac{mM}{r^2}, \tag{1}$$

where $G \approx 6.67 \times 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$ is the gravitational constant (first measured by Cavendish).

The force is apparently *conservative* and allows potential formulation: the body M creates a (non-relativistic) gravitational potential ϕ,

$$\phi(\vec{r}) = -\frac{GM}{r}, \tag{2}$$

in which the particle m acquires a potential energy $m\phi$ with the corresponding force

$$\vec{F} = -m\nabla\phi. \tag{3}$$

The potential (2) satisfies the Poisson equation[1]

$$\nabla^2\phi(\vec{r}) = 4\pi GM\delta(\vec{r}). \tag{4}$$

If instead of a single mass with density $M\delta(\vec{r})$ there is a distribution of masses with density $\mu(\vec{r})$ the gravitational potential created by these masses apparently satisfies the Poisson equation

$$\nabla^2\phi = 4\pi G\mu. \tag{5}$$

Equation (3) can be cast into a variational form with the action

$$\begin{aligned} S &= \int dt\left(\frac{1}{2}mv^2 - m\phi - mc^2\right) \\ &= -mc\int dt\left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right). \end{aligned} \tag{6}$$

Comparing with $S = -mc\int ds$ we get (squaring and dropping terms negligible in the limit $c \to \infty$)

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2dt^2 - d\vec{r}^2. \tag{7}$$

Thus in the Newtonian limit the metric tensor can be approximated[2] by $g_{ab} = \eta_{ab} + h_{ab}$, where η_{ab} is the Minkowski metric tensor and h_{ab} is a small correction, and the g_{00} component is given as

$$g_{00} = 1 + \frac{2\phi}{c^2}, \tag{8}$$

where ϕ satisfies the Poisson equation (5).

[1] $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$.

[2] where we have neglected the terms $g_{\alpha\beta},\ \alpha\beta = 1,2,3$ since their contribution to ds^2 is not multiplied by c^2 and is thus negligible compared to the contribution from g_{00}.

Newtonian limit of general relativity

For a distribution of (otherwise non-interacting) masses with mass density $\mu(\vec{r})$ the energy-momentum tensor is

$$T_{ab} = \mu u_a u_b. \tag{9}$$

In the Newtonian limit, where all fields are weak and all velocities are small, $u_a = \{1,0,0,0\}$, only the $_{00}$ component of the energy-momentum tensor is non-vanishing,

$$T_{00} = \mu \tag{10}$$

Therefore we shall only consider the $_{00}$ component of the Einstein's equation,

$$R_{00} = \kappa(T_{00} - \frac{1}{2}g_{00}T) = \frac{1}{2}\kappa\mu. \tag{11}$$

In the slow-weak limit all second order terms and temporal derivatives must be neglected altogether. The $_{00}$ component of the Ricci tensor then reduces to

$$R_{00} \doteq R^a_{0a0} = \Gamma^\alpha_{00,\alpha} \tag{12}$$

where the Greek symbols run over $1,2,3$.

Assuming $g_{00} = 1 + 2\phi$ and dropping the temporal derivatives, the Christoffel symbol becomes

$$\Gamma^\alpha_{00} = -\phi^{,\alpha}, \tag{13}$$

The Ricci tensor in the same limit is given as

$$R_{00} = -\phi^{;\alpha}_{;\alpha} \equiv \nabla^2\phi, \tag{14}$$

The Einstein equation thus turns into the Poisson's equation

$$\nabla^2\phi = \frac{1}{2}\kappa\mu \tag{15}$$

which is equivalent to the Newtonian theory if we put

$$\kappa = \frac{8\pi G}{c^4}. \tag{16}$$

Gravitational waves

In a weak gravitational field the space-time is almost flat and the metric tensor g_{ab} is equal to the flat metric η_{ab} plus a small correction h_{ab},

$$g_{ab} \;=\; \eta_{ab} + h_{ab}\,. \tag{17}$$

The Riemann tensor to the lowest order in h_{ab} is

$$R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}). \tag{18}$$

If we choose coordinates such that

$$(h^a_b - \frac{1}{2}h\delta^a_b)^{,b} = 0, \tag{19}$$

the Ricci tensor is simply

$$R_{ab} = -\frac{1}{2}h^{,c}_{ab,c} \tag{20}$$

and the vacuum Einstein's equations turn into the ordinary wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) h_{ab} = 0\,.$$

The intensity of gravitational radiation by a system of slowly moving bodies is determined by its quadrupole moment $D_{\alpha\beta}$

$$-\frac{dE}{dt} = \frac{G}{45c^5}(D'''_{\alpha\beta})^2 \tag{21}$$

Exercises

1. Calculate the energy-momentum tensor T_{ab} for a particle of mass m with the action $S = -m\int ds$. Hint: calculate the variation of the action with respect to δg_{ab} and represent it in the form $\delta S = -\frac{1}{2}\int T^{ab}\delta g_{ab} ds$.

2. Dirac's delta-function $\delta(\vec{r})$ is zero everywhere except for the origin, where it is infinitely large, such that

$$\int d^3r\delta(\vec{r}) = 1\,.$$

   Show that $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$

3. (**Obligatory**) Show that from the metric (7),

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2dt^2 - d\vec{r}^2\,,$$

   it follows, that time runs differently at different places in a gravitational potential and estimate the difference in the clock rates at the sea level and on top of the Everest mountain.