Riemann curvature tensor

Christoffel symbols are not tensors and cannot be (directly) used in variational principles. In this section we shall construct a covariant tensor, called Riemann curvature tensor, associated with the curvature of the space, which can be used in building a covariant action of gravitation.

Under a parallel transport (that is, whith $DA^a = 0$) of a vector A^a along a path dx, the components of the vector (generally) change,

$$dA^a\big|_{DA^a=0} = -\Gamma^a_{bc} A^b dx^c \,. \qquad (1)$$

The change of a vector along an infinitesimal closed contour, $\Delta A^a = \oint dA^a$, is a covariant quantity as it is the difference of two vectors at the same point. It is apparently proportional to the vector itself and the area of the surface enclosed by the contour.

Let us calculate this change using a simple rectangular contour with sides dx and dx'. The vector A^a after a parallel transform along dx and then dx' turns into

$$A^a(x+dx+dx') = A^a - \Gamma^a_{bc} A^b dx^c \\ -\left(\Gamma^a_{bc} + \Gamma^a_{bc,d} dx^d\right)\left(A^b - \Gamma^b_{df} A^d dx^f\right) dx'^c \,. \qquad (2)$$

Similarly, after a parallel transform first along dx' and then along dx

$$A^a(x+dx'+dx) = A^a - \Gamma^a_{bc} A^b dx'^c \\ -\left(\Gamma^a_{bc} + \Gamma^a_{bc,d} dx'^d\right)\left(A^b - \Gamma^b_{df} A^d dx'^f\right) dx^c \,. \qquad (3)$$

Subtracting these two and leaving only the lowest order terms gives (after some index renaming)

$$\Delta A^a \doteq A^a(x+dx'+dx) - A^a(x+dx+dx') \\ = \left(\Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{fc}\Gamma^f_{bd} - \Gamma^a_{fd}\Gamma^f_{bc}\right) A^b dx^c dx'^d \\ \equiv R^a_{bcd} A^b dx^c dx'^d \,. \qquad (4)$$

The tensor in brackets is called the Riemann curvature tensor R^a_{bcd},

$$R^a_{bcd} \doteq \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{fc}\Gamma^f_{bd} - \Gamma^a_{fd}\Gamma^f_{bc}. \qquad (5)$$

It follows from (4) that if the Riemann tensor vanishes, the vector A^a does not change if parallelly transported along an arbitrary infinitesimal closed path. Also after a parallel transform from x to $x+\delta x$ the vector $A^a(x+\delta x)$ is independent on which path is taken. The space where Riemann tensor vanishes everywhere is called flat.

The Riemann tensor defines also the commutator of covariant derivatives

$$(D_a D_b - D_b D_a) A_c = R^d_{abc} A_d, \qquad (6)$$

where $D_a \equiv \frac{D}{dx^a}$.

Properties of the curvature tensor

1. Antisymmetry:

$$R_{abcd} = -R_{bacd} = -R_{abdc} \,. \qquad (7)$$

2. Interchange symmetry:

$$R_{abcd} = R_{cdab} \,. \qquad (8)$$

3. Algebraic Bianchi identity:

$$R_{a[bcd]} \doteq R_{abcd} + R_{acdb} + R_{adbc} = 0 \,. \qquad (9)$$

4. Differential Bianchi identity:

$$R_{ab[cd;e]} = 0, \qquad (10)$$

Ricci tensor and scalar

The Ricci tensor is obtained by a contraction of the Riemann tensor over pair of indexes,

$$R_{ab} \doteq R^d_{adb} \,. \qquad (11)$$

It is symmetric,

$$R_{ab} = R_{ba} \,. \qquad (12)$$

The Ricci scalar (or scalar curvature) is the simplest curvature scalar in a Riemannian geometry,

$$R \doteq g^{ab} R_{ab} \,. \qquad (13)$$

Contraction of the differential Bianchi identity over bc and ad gives

$$R^b_{a;b} = \frac{1}{2} R_{,a} \,, \qquad (14)$$

from which it follows, that the (Einstein) tensor

$$G_{ab} \doteq R_{ab} - \frac{1}{2} R g_{ab} \qquad (15)$$

is divergenceless

$$G^{ab}_{;b} = 0 \,. \qquad (16)$$

Hilbert action and Einstein equation

The simplest action for gravitation is the Hilbert action – an integral over the Ricci scalar,

$$S_g = -\frac{1}{2\kappa}\int R\sqrt{-g}d\Omega, \tag{17}$$

where κ is a constant (to be determined from the experiment).

Variation of $R\sqrt{-g}$ with respect to δg^{ab} gives

$$\delta\left(R\sqrt{-g}\right) = \delta\left(R_{ab}g^{ab}\sqrt{-g}\right)$$
$$= R_{ab}\delta g^{ab}\sqrt{-g} + R\delta\sqrt{-g} + g^{ab}\sqrt{-g}\delta R_{ab}. \tag{18}$$

The last term can be proved (see t'Hooft) to not contribute to variation; in the second term we have[1]

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}gg^{ab}\delta g_{ab} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}. \tag{19}$$

Thus the variation of the Hilbert action is

$$\delta S_g = -\frac{1}{2\kappa}\int\left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\delta g^{ab}\sqrt{-g}d\Omega. \tag{20}$$

As we have shown above, the variation of the matter action with respect to the metric tensor is determined by the energy-momentum tensor,

$$\delta S_m = \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega\,.$$

Combining the variations of the matter action and the gravitational action into the variational principle

$$\delta(S_m + S_g) = 0 \tag{21}$$

gives the famous Einstein's gravitational field equation,

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}\,. \tag{22}$$

Note that the covariant divergence of the left-hand side vanishes, due to Bianchi identity. Therefore the covariant divergence of the right-hand side, the energy-momentum tensor, should also vanish (and it apparently does so as we proved above). The latter, however, contains the equations of motion for the matter. Therefore the Einstein equation determines simultaneously both the gravitational field created by matter and the motion of the matter in this gravitational field.

[1] $dg = gg^{ab}dg_{ab} = -gg_{ab}dg^{ab}$.

Exercises

1. Prove the differential Bianchi identity (e.g. using the locally flat system where Christoffel symbols (but not their derivatives) are equal zero).
2. Prove that $(R^{ab} - \frac{1}{2}Rg^{ab})_{;a} = 0$.
3. (Obligatory) Compute all the non-vanishing components of the Riemann tensor R_{abcd} (where $a, b, c, d = \theta, \phi$) for the metric

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{23}$$

on a 2-dimensional sphere of radius r. Calculate also the Ricci tensor R_{ab} and the scalar curvature (Ricci scalar) R.

Answer:

$$R_{\theta\phi\theta\phi} = r^2\sin^2\theta = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi}$$