Matter action in curved space

In general relativity gravitation is the geometry of the curved space-time (the metric tensor) rather than a matter field like the electromagnetic field. The action of the matter[1] has the same form as in Minkowski space only written in a generally covariant way. The matter then couples to the gravitational field through the metric tensors in the matter action.

Covariant volume element. The volume element $d\Omega \equiv d^4x$ is not invariant under a general coordinate transformation. In curved spaces it has to be substituted with a covariant expression, $\sqrt{-g}d\Omega$, where g is the determinant of the metric tensor g_{ab} $(g<0)$.

Indeed the metric tensor transforms as

$$g_{ab} = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g'_{cd}. \tag{1}$$

Taking determinant of both sides gives $g = J'^2 g'$, or

$$\sqrt{-g} = J'\sqrt{-g'} \tag{2}$$

where $J' = \left|\frac{\partial x'^a}{\partial x^b}\right|$ is the Jacobian of the transformation. The 4-volume transforms as

$$d\Omega = \left|\frac{\partial x^a}{\partial x'^b}\right| d\Omega' = \frac{1}{J'}d\Omega'. \tag{3}$$

Apparently the combination $\sqrt{-g}d\Omega$ transforms as

$$\sqrt{-g}d\Omega = J'\sqrt{-g'}\frac{1}{J'}d\Omega' = \sqrt{-g'}d\Omega', \tag{4}$$

and is thus a covariant volume element.

Matter action. The action of the matter in general relativity has the same form as in special relativity, only rewritten, if needed, in a generally covariant way. Particularly, $d\Omega \to \sqrt{-g}d\Omega$, $\partial^a\varphi \to g^{ab}\partial_b\varphi$, and $\partial_a A^b \to D_a A^b$. For example,

$$\int A_a j^a d\Omega \quad\to\quad \int A_a j^a \sqrt{-g}d\Omega\,, \tag{5}$$

$$\int \partial^a\varphi\partial_a\varphi d\Omega \quad\to\quad \int g^{ab}\partial_a\varphi\partial_b\varphi\sqrt{-g}d\Omega\,, \tag{6}$$

$$\int F^{ab}F_{ab}d\Omega \quad\to\quad \int F^{ab}F_{ab}\sqrt{-g}d\Omega \tag{7}$$

[1] *matter* is all fields other than gravitational.

Energy-momentum tensor of matter

The variation of the matter action,

$$S_m = \int \mathcal{L}\sqrt{-g}d\Omega, \tag{8}$$

under the variation δg^{ab} can be written in terms of a symmetric tensor T_{ab},

$$\begin{aligned}\delta S_m &\doteq \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega \\ &= -\frac{1}{2}\int T^{ab}\delta g_{ab}\sqrt{-g}d\Omega,\end{aligned} \tag{9}$$

where[2]

$$\frac{1}{2}\sqrt{-g}T_{ab}\delta g^{ab} \doteq \delta(\sqrt{-g}\mathcal{L})\,. \tag{10}$$

The tensor T_{ab} is actually the energy-momentum tensor, since in a flat space it satisfies a conservation law. Indeed, consider an infinitesimal coordinate transformation,

$$x^a \to x'^a = x^a + \epsilon^a\,. \tag{11}$$

The variation of the metric tensor under this transformation can be written as

$$\delta g^{ab} = \epsilon^{a;b} + \epsilon^{b;a}\,, \delta g_{ab} = -\epsilon_{a;b} - \epsilon_{b;a}\,. \tag{12}$$

The variation of the action then takes the form

$$\delta S = \int T_{ab}\epsilon^{a;b}\sqrt{-g}d\Omega\,. \tag{13}$$

Integrating by parts[3],

$$\delta S = -\int T^b_{a;b}\epsilon^a\sqrt{-g}d\Omega \tag{14}$$

Thus the tensor T^a_b satisfies the equation

$$T^{ab}_{;b} = 0\,, \tag{15}$$

which in a flat space turns into the energy-momentum conservation equation $T^{ab}_{,b} = 0$. One can thus assume that the tensor is proportional to the canonical energy-momentum tensor. Direct calculations show that the proportionality factor is equal unity.

[2] From $g_{ab}g^{bc} = \delta^c_a$ follows $g_{ab}\delta g^{bc} = -\delta g_{ab}g^{bc}$ and therefore $T_{ab}\delta g^{ab} = -T^{ab}\delta g_{ab}$.

[3] the total differential does not contribute, as usual.

Exercises[4]

1. In a curved space the electromagnetic field strength tensor F_{ab} is defined as $F_{ab} = A_{b;a} - A_{a;b}$ and the first Maxwell equation is $F_{ab;c} + F_{bc;a} + F_{ca;b} = 0$. Show that in the torsion free space of general relativity, $\Gamma^a_{bc} = \Gamma^a_{cb}$, these equations can still be written as in Minkowski space, $F_{ab} = A_{b,a} - A_{a,b}$ and $F_{ab,c} + F_{bc,a} + F_{ca,b} = 0$.

2. (**Obligatory**) Derive the second Maxwell equation in a curved space,
$$\left(\sqrt{-g}F^{ab}\right)_{,a} = 4\pi\sqrt{-g}j^b \ ,$$
from the action
$$S = \int \left(-\frac{1}{16\pi}F^{ab}F_{ab} - A_a j^a\right)\sqrt{-g}d\Omega\,.$$
Show that the equation can also be written as
$$F^{ab}_{;a} = 4\pi j^b\,.$$
Hints:
   (a) show that $\Gamma^a_{ba} = \frac{1}{2g}g_{,b} = (\ln\sqrt{-g})_{,b}$
   (b) show that $F^{ab}_{;a} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ab})_{,a}$

3. The Lagrangian density for the electromagnetic field is
$$\mathcal{L} = -\frac{1}{16\pi}F_{ab}F^{ab}.$$
Calculate the corresponding energy-momentum tensor using $\frac{1}{2}\sqrt{-g}T_{ab} = \frac{\partial\sqrt{-g}\mathcal{L}}{\partial g^{ab}}$.
Answer: $T_{ab} = \frac{1}{4\pi}(-F_{ac}F^c_b + \frac{1}{4}F_{cd}F^{cd}g_{ab})$

4. (**Obligatory**) In the Minkowski space consider a scalar field φ with action
$$S = \int d\Omega\left(-\frac{1}{2}\varphi^{,a}\varphi_{,a} - \frac{1}{2}m^2\varphi^2\right)$$
and calculate its "translation-invariance"energy-momentum tensor,
$$T^a_b = \frac{\partial\mathcal{L}}{\partial\varphi_{,a}}\varphi_{,b} - \mathcal{L}\delta^a_b\ .$$
Rewrite the action in a generally covariant form and calculate its "metric"energy-momentum tensor,
$$\frac{1}{2}\sqrt{-g}T_{ab} = \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{ab}}\ .$$

[4] notation: $_{;a} \equiv D_a \equiv \frac{D}{dx^a}$ and $_{,a} \equiv \partial_a \equiv \frac{\partial}{dx^a}$