Variational principle

Action and variational principle

Action is a functional which takes the trajectory of the system (also called *path* or *history*) as its argument and returns a covariant real scalar as the result.

Generally, the action takes different values for different paths. Classical mechanics postulates that

> *the path actually followed by a physical system is that for which the variation of the action vanishes,*
>
> $$\delta S = 0. \quad (1)$$

Vanishing variation means the action is stationary: it has an extremum – generally speaking, minimum.

The postulate is called *variational* or *least action* principle.

The classical equations of motion can be derived from the variational principle.

If the action is represented as an integral over time, taken along the path of the system between the initial time and the final time of the development of the system,

$$S = \int L dt \,, \quad (2)$$

the integrand L is called the *Lagrangian*.

Examples

Newton's equation

For a non-relativistic body with mass m and coordinates $\vec{r}$, moving in an external potential $V(\vec{r})$, the action is

$$S = \int \left(\frac{m\vec{v}^2}{2} - V(\vec{r}) \right) dt, \quad (3)$$

where $\vec{v} = d\vec{r}/dt$ is the velocity of the body.

Variation of the action under infinitesimal variation of the trajectory $\vec{r}(t) \to \vec{r}(t) + \delta\vec{r}(t)$ is[1]

$$\delta S = \int \left(m\vec{v}\delta\vec{v} - \vec{\nabla} V \delta\vec{r} \right) dt. \quad (4)$$

Integrating the first term by parts gives

$$\delta S = \int \left(-m\frac{d\vec{v}}{dt} - \vec{\nabla} V \right) \delta\vec{r} dt. \quad (5)$$

Since the variation $\delta\vec{r}$ is arbitrary, the expression in parentheses has to be equal zero at the stationary trajectory, which gives the Newton's equation of motion

$$m\frac{d\vec{v}}{dt} = -\vec{\nabla} V \,. \quad (6)$$

Lorentz force

In classical electrodynamics (where the space is Minkowski) the action of a body with mass m and charge e moving in a given electromagnetic field A^a is given as

$$S = -m \int ds - e \int A_a dx^a. \quad (7)$$

A small variation of the trajectory of the body, $x^a \to x^a + \delta x^a$, leads to the following variation of the action[2],

$$\delta S = \int \left(-m\delta dx^a u_a - eA_a \delta dx^a - e dx^a A_{a,b} \delta x^b \right) . \quad (8)$$

Integrating the first and the second terms by parts (and renaming indexes in the third term) gives[3]

$$\delta S = \int ds \delta x^a \left(m\frac{du_a}{ds} + eA_{a,b}u^b - eu^b A_{b,a} \right) . \quad (9)$$

Since the variation δx^a is arbitrary, the expression in parentheses has to vanish identically on physical trajectories, giving the the *Lorentz force* equation,

$$m\frac{du_a}{ds} = eF_{ab}u^b \quad (10)$$

where

$$F_{ab} = A_{b,a} - A_{a,b} \quad (11)$$

is the electromagnetic field tensor.

[1] the *nabla* operator is defined as

$$\vec{\nabla} \equiv \sum_{\alpha=1}^{3} \vec{e}_\alpha \frac{\partial}{\partial x_\alpha}$$

[2] in a flat space we don't have to worry about derivatives of the metric tensor.

[3] as always $\frac{\partial}{\partial x^a} \equiv_{,a} \equiv \partial_a$.

Maxwell equations

In classical electrodynamics the action for the electromagnetic field A^a with given sources j^a is written as an integral over the 4-volume,

$$S = -\frac{1}{8\pi}\int d^4x\, A_{a,b}A^{a,b} - \int d^4x\, A_a j^a\,. \qquad (12)$$

Canonical calculation of the variation of the action under infinitesimal variation of the field $A^a \to A^a + \delta A^a$ gives the second Maxwell equation,

$$A^{a,b}_{,b} = 4\pi j^a\,. \qquad (13)$$

Exercises

1. Rewrite the covariant Lorentz force equation in 3-notation, where

$$\begin{aligned} A^a &= \{\phi, \vec{A}\} \\ \vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\,, \\ \vec{H} &= \mathbf{rot}\vec{A} \equiv \vec{\nabla}\times\vec{A}\,. \end{aligned}$$

2. (**Obligatory**) In Minkowski space

   (a) derive[4] the Maxwell equation with sources,

   $$\partial_a\partial^a A^b = 4\pi j^b\,,$$

   from the action

   $$S = -\frac{1}{8\pi}\int d^4x A_{a,b}A^{a,b} - \int d^4x A^a j_a.$$

   (b) show that with the Lorenz condition,

   $$A^a_{,a} = 0\,,$$

   it is equivalent to

   $$F^{ab}_{,a} = 4\pi j^b\,.$$

[4]using the Gauss theorem

$$\int_{\mathrm{V}} d^4x\,\frac{\partial A^a}{\partial x^a} = \oint_{\partial\mathrm{V}} A^a dS_a$$

where dS_a is an infinitesimal element of the hyper-surface.