Geodesic

The term *geodesic* comes from *geodesy*, the science of measuring the size and shape of Earth. In the original sense, a geodesic was the shortest route between two points on the Earth's surface, namely, a segment of a great circle. The term has since been generalised to include measurements in more general mathematical spaces.

Geodesic as no-acceleration trajectory

The velocity vector u^a of a body moving along a trajectory is defined as

$$u^a = \frac{dx^a}{ds} . \tag{1}$$

where dx^a is the infinitesimal vector along the trajectory and ds is the invariant interval.

A free body in curvilinear coordinates moves in such a way that the covariant derivative of its 4-velocity vanishes,

$$Du^a = 0 , \tag{2}$$

which is called the *geodesic* equation. It can also be written as

$$\frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0 , \tag{3}$$

or

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0. \tag{4}$$

Geodesic as shortest route

The invariant "length" of a trajectory of a moving body is defined as the sum of infinitesimal intervals ds along the trajectory,

$$S = \int ds . \tag{5}$$

The shortest trajectory is the one where the variation of the length as function of the trajectories vanishes,

$$\delta S = 0. \tag{6}$$

To calculate the variation of the length we first vary the square interval $ds^2 = g_{ab} dx^a dx^b$,

$$2ds\delta ds = \delta g_{ab} dx^a dx^b + 2g_{ab} \delta dx^a dx^b , \tag{7}$$

which gives the variation of ds,

$$\delta ds = \frac{1}{2} \delta g_{ab} u^a u^b ds + d\delta x^a u_a . \tag{8}$$

The second term should be integrated by parts using

$$d\delta x^a u_a = d(\delta x^a u_a) - \delta x^c du_c. \tag{9}$$

The full differential does not contribute to the variation, and we finally arrive at[1]

$$\delta S = \int ds \delta x^c \left(\frac{du_c}{ds} - \frac{1}{2} g_{ab,c} u^a u^b \right) = 0 \tag{10}$$

Since the variation δx is arbitrary, it is the expression in brackets that should be equal zero, which gives the equation of motion

$$\frac{du_c}{ds} - \frac{1}{2} g_{ab,c} u^a u^b = 0. \tag{11}$$

which is equivalent to the no-acceleration equation (4).

Motion of free bodies in general relativity

In general relativity a free body (that is, not affected by physical forces) moves along a geodesic. Massive bodies do not create physical fields around them but rather distort space-time in their vicinity causing the geodesics to become "curved".

Exercises

1. Prove that (3) and (11) are equivalent.

2. (Obligatory) Consider the parametric equations for a straight line in Cartesian coordinates x and y,

   $$\frac{d^2x}{ds^2} = 0 , \; \frac{d^2y}{ds^2} = 0. \tag{12}$$

   Make a coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$ and obtain the corresponding equations in the r, θ coordinates. Prove that they are identical to geodesic equations (4).

[1] $f_{,a} \equiv \frac{\partial f}{\partial x^a}$