Curvilinear coordinates

Flat and curved spaces

A space with Euclidean or pseudo-Euclidean metric is called *flat* and the coordinates in which the metric is (pseudo) Euclidean is also often called flat. For example, the Minkowski space of special relativity is flat.

If the metric is not everywhere (pseudo) Euclidean, the coordinates are called curvilinear. For example, in the flat 2-dimensional space with polar coordinates $\{r, \theta\}$ the metric is non-Euclidean,

$$dl^2 = dr^2 + r^2 d\theta^2 . \tag{1}$$

However, if there exist a coordinate transformation that globally (that is, everywhere) turns the metric into (pseudo) Euclidean, the space is still called flat. In our example such transformation is

$$\begin{cases} x = r\cos\theta\,, \\ y = r\sin\theta\,, \end{cases} \tag{2}$$

If such transformation does not exist, the space is called *curved* or *Riemann space*. The geometry of a curved space is called non-Euclidean geometry or *Riemann geometry*.

Space-time in a gravitational field

According to Einstein's equivalent principle, gravitational forces disappear (locally) in the frame of a free falling observer. A free falling observer then finds themselves in a Minkowski space-time.

In other words the space-time in a gravitational field can be reduced locally to Minkowski space-time by a (non-linear) coordinate transformation.

Yet in other words the space-time in a gravitational field can be obtained by a (non-linear) coordinate transformation from a Minkowski space-time. This transformation cannot be global, since gravitational forces, unlike inertial forces, vanish at large distances from massive bodies. Thus the space-time in a gravitational field is curved.

General principle of relativity

Since in the presence of gravitational fields the space-time is curved, and it is not possible to build a set of globally flat coordinates, any frame of reference with arbitrary curvilinear coordinates (and arbitrarily tuned clocks) must be equally accepted in general relativity[1].

The principle of general relativity is then formulated as

> The laws of physics should have the same form in arbitrary frames of reference.

In order to build differential equations, invariant under general coordinate transformations, one needs to develop differential geometry in curvilinear coordinates.

Covariant differentiation

Curvilinear coordinates

A set of four numbers (for example *time*, *latitude*, *longitude*, an *height* above the sea level) used to specify the location of an event in space-time is called (curvilinear) coordinates and is denoted as $\{x^0, x^1, x^2, x^3\}$, or as x^a, where $a = 0, 1, 2, 3$, or simply as x where it does not lead to confusion[2].

Under a general (non-linear) coordinate transformation $x \to x'$ the coordinate differentials dx^a transform linearly through the Jacobian matrix, while partial derivatives of a scalar function $\phi(x)$ transform also linearly though through the inverse Jacobian matrix,

$$dx^a = \sum_{b=0}^{3} \frac{\partial x^a}{\partial x'^b} dx'^b \,,\; \frac{\partial \phi}{\partial x^a} = \sum_{b=0}^{3} \frac{\partial \phi}{\partial x'^b} \frac{\partial x'^b}{\partial x^a} \,. \tag{3}$$

Covariant vectors and metric tensor

A set of four quantities A^a, where $a = 0, 1, 2, 3$, is called a *contravariant vector* if under a coordinate transformation it transforms as coordinate differentials dx^a, while a set of four quantities A_a is called a *covariant vector* if it transforms[3] as derivatives of a scalar $\partial\phi/\partial x^a$,

$$A^a = \frac{\partial x^a}{\partial x'^b} A'^b ,\; A_a = \frac{\partial x'^b}{\partial x^a} A'_b . \tag{4}$$

Quantities which transform as products of vectors (either co- or contra-variant) are said to trans-

[1] The only condition is that the coordinates are differentiable to allow development of differential geometry.

[2] Typically, where possible the x^0 coordinate is time.

[3] In the following $\boxed{A^a B_a \equiv \sum_{a=0}^{3} A^a B_a}$

form *covariantly*. These quantities are called *tensors* with the number of the vector indices called the *order* of the tensor.

The contraction $A^a B_a$ is apparently a scalar (zeroth order tensor), as it is invariant under coordinate transformation, $A^a B_a = A'^a B'_a$.

The *metric tensor* g_{ab} defines the invariant infinitesimal element ds^2 in curvilinear coordinates,

$$ds^2 = g_{ab} dx^a dx^b. \tag{5}$$

Since dx^a is an arbitrary contra-variant vector, the construction $g_{ab} dx^b$ transforms as a co-variant vector and thus the metric tensor connects contra- and co-variant vectors,

$$A_a = g_{ab} A^b \,. \tag{6}$$

Covariant differential and Christoffel symbol

In curvilinear coordinates the differential of a vector dA^a is not a covariant quantity, since generally

$$dA_a = d(g_{ab} A^b) \neq g_{ab} dA^b. \tag{7}$$

The *covariant differential*, DA^a, contains an additional contribution, which ensures covariance,

$$DA^a = dA^a + \Gamma^a_{bc} A^b dx^c \,, \tag{8}$$

where Γ^a_{bc} is called the *Christoffel symbol*.

Differential of a scalar $d(A_a B^b)$ is already a covariant quantity, therefore $D(A_a B^a) = d(A_a B^b)$ for an arbitrary B^a, which leads to

$$DA_a = dA_a - \Gamma^b_{ac} A_b dx^c \,. \tag{9}$$

Using the "comma" notation for partial derivatives,

$$dA^a = \frac{\partial A^a}{\partial x^c} dx^c \equiv A^a_{,c} dx^c \,, \tag{10}$$

the covariant differentials take the form

$$DA^a = \left(A^a_{,c} + \Gamma^a_{bc} A^b\right) dx^c \,, \tag{11}$$

$$DA_a = \left(A_{a,c} - \Gamma^b_{ac} A_b\right) dx^c \,. \tag{12}$$

The expressions in the parentheses are tensors, since when multiplied by arbitrary vector dx^c they produce vectors. These tensors are called *covariant derivatives* and are denoted as

$$A^a_{;c} = A^a_{,c} + \Gamma^a_{bc} A^b \,, \tag{13}$$

$$A_{a;c} = A_{a,c} - \Gamma^b_{ac} A_b \,. \tag{14}$$

Consider the difference,

$$\phi_{;a;c} - \phi_{;c;a} = (\Gamma^b_{ca} - \Gamma^b_{ac}) \phi_{;b} \,, \tag{15}$$

where ϕ is a scalar. The left-hand side is a tensor, therefore $\Gamma^b_{ca} - \Gamma^b_{ac}$ is also a tensor. The latter is equal zero in a locally flat frame. Being a tensor it is than equal zero in all other frames. Therefore Christoffel symbol is symmetric over exchange of the two lower indexes,

$$\Gamma^b_{ca} = \Gamma^b_{ac} \,. \tag{16}$$

For DA_a to be a covariant quantity one needs

$$DA_a = g_{ab} DA^b = D g_{ab} A^b, \tag{17}$$

i.e. the covariant derivative of the metric tensor[4] has to vanish, $Dg_{ab} = 0$, which defines the Christoffel symbols,

$$\Gamma_{abc} = \frac{1}{2} \left(g_{ab,c} - g_{bc,a} + g_{ac,b}\right). \tag{18}$$

Exercises

1. (Obligatory) Let x, y be Cartesian coordinates in a flat two-dimensional space with metric $dl^2 = dx^2 + dy^2$. Consider polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ (another notation: $x^r \equiv r$, $x^\theta \equiv \theta$)
   (a) Calculate the interval dl^2 in polar coordinates and find the corresponding metric tensor g_{ab} (where indexes a and b take the values r, θ).
   (b) From[5] $d\vec{l} = dr\vec{e}_r + d\theta\vec{e}_\theta$ find $\vec{e}_a$, $a = r, \theta$.
   (c) Check that $\vec{e}_a \cdot \vec{e}_b = g_{ab}$.
   (d) Find $g^{ab} = g^{-1}_{ab}$, $\vec{e}^a = g^{ab}\vec{e}_b$, and $\vec{e}^a \cdot \vec{e}_b$.
   (e) Consider vector $\vec{A} = A^a \vec{e}_a = A_a \vec{e}^a$. Prove that[6]
   $$DA^a \equiv \vec{e}^a \cdot d\vec{A} = dA^a + (\vec{e}^a \cdot \vec{e}_{b,c})\, A^b dx^c,$$
   $$DA_a \equiv \vec{e}_a \cdot d\vec{A} = dA_a - \left(\vec{e}^b \cdot \vec{e}_{a,c}\right) A_b dx^c.$$
   Prove that the expression in parentheses is the Christoffel symbol (as in eq. (18)).

[4] Considering $D(A^a B^b)$ one can define the covariant differential of a tensor,
$$DF^{ab} = dF^{ab} + \Gamma^a_{cd} F^{cb} dx^d + \Gamma^b_{cd} F^{ac} dx^d.$$

[5] $d\vec{l} \equiv dx\vec{e}_x + dy\vec{e}_y$, where $\vec{e}_x \cdot \vec{e}_x = \vec{e}_y \cdot \vec{e}_y = 1$, $\vec{e}_x \cdot \vec{e}_y = 0$.

[6] comma-index denotes partial derivative: $\phi_{,c} \equiv \frac{\partial \phi}{\partial x^c}$.