Special relativity

Einstein's special relativity is a theory of spatial and temporal measurements in inertial frames of reference, formulated by Albert Einstein in 1905. It is the basis of relativistic mechanics. In the slow motion limit special relativity reduces to Galilean relativity.

Postulates

Special relativity is based on several postulates, which are deduced from a number of experiments:

1. **Existence of inertial frames**: there exist inertial frames of reference, which are moving with constant velocities with respect to each other, with Cartesian coordinates, where the laws of physics take their simplest form. In particular, free bodies (that is, not affected by forces) move with constant velocities along lines (straight curves).

2. **Special principle of relativity**: all inertial frames are equivalent, hence the laws of physics must have the same form in all inertial frames.

3. **Finiteness of the speed of light**: the highest velocity for a physical object, the speed of light in vacuum, is finite (and actually relatively small, 299792458m/s).

Lorentz and Galilean transformations

Let us consider linear transformations between inertial frames with parallel Cartesian coordinates moving with relative velocity v along one of the axes[1]. The general form of such transformation, consistent with the principle of relativity (and also isotropy of space), has the form (see exercise 1)

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & -v/c^2 \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ z \end{pmatrix}, \quad (1)$$

where the frame with coordinates (t', z') moves relative to the frame with coordinates (t, z) with velocity v along the z (and z') axis.

[1]this transformation is often called *Lorentz boost*, or *velocity boost*, or simply *boost*.

The x- and y-coordinates, perpendicular to the velocity boost, transform identically and are therefore omitted for brevity.

The velocity c is a universal constant, the fastest possible relative velocity of two inertial frames. Velocity c is experimentally measured to be finite (and actually relatively small, 299792458 m/s).

Transformation (1) with finite c is called the Lorentz transformation. Note that time and space do not transform separately but rather as components of one inseparable four-component space-time point,

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \quad (2)$$

In the limit $c \to \infty$ the Lorentz transformation turns into Galilean transformation,

$$\begin{aligned} t' &= t\,, \\ z' &= z - vt\,. \end{aligned} \quad (3)$$

Here time is absolute and does not transform at all. The time-space coordinates then separate into invariant time and three spatial coordinates.

Invariant interval and metric

Direct calculation shows that the *interval*

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad (4)$$

is invariant under Lorentz transformation (1) and thus defines a *metric*[2]. A space with a metric is called *metric space*.

The pseudo-Euclidean[3] metric (4) is called *Minkowski metric* and a space with such metric is called *Minkowski space*.

The existence of a metric allows developement of a geometry of space, that is, measurements of

[2]*Metric* is a function of two infinitesimally close points in a space, which is used to measure distances and angles (that is, to develop a geometry of a space).

[3]*Euclidean metric* in an n-dimensional space has the form

$$ds^2 = dx_1^2 + \cdots + dx_n^2,$$

while *pseudo-Euclidean metric* has one or more negative signs,

$$ds^2 = dx_1^2 + \cdots + dx_k^2 - dx_{k+1}^2 - \cdots - dx_n^2.$$

distances, angles, and time intervals. However, geometry in Minkowski space, called Minkowski geometry, is different from the everyday Euclidean geometry. In particular, distances and time intervals are relative, that is, different in different inertial frames.

In the limit $v \ll c$ it reduces to *Euclidean space*, which is the non-relativistic world of classical mechanics with Galilean transformation, where dt is itself invariant and the Minkowski metric reduces to the Euclidean metric,

$$dl^2 = dx^2 + dy^2 + dz^2 \tag{5}$$

Exercises

1. (Obligatory) Derive the Lorentz transformation matrix,

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & -v/c^2 \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ z \end{pmatrix}, \tag{6}$$

using e.g. the following strategy:

   (a) Argue, that a transformation between inertial frames is a linear transformation,

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \Lambda(v) \begin{pmatrix} t \\ z \end{pmatrix},$$

   and assume the following form of the transformation matrix Λ,

$$\Lambda = \begin{bmatrix} \gamma & \delta \\ \beta & \alpha \end{bmatrix}.$$

   (b) Consider the motion of the origin of the frame K' (K) relative to frame K (K') and show that

$$\beta = -v\gamma\,,\ \beta = -v\alpha\,.$$

   (c) Consider the inverse transformation and argue that matrices $\Lambda(v)^{-1}$ and $\Lambda(-v)$ should be equal; argue that in an isotropic space $\gamma(v) = \gamma(-v)$; show that this gives

$$\gamma^2 + v\gamma\delta = 1\,.$$

   (d) Consider a composition of two transformations, $\Lambda(v)\Lambda(v')$, and argue that this should be equal to a lambda-matrix with certain velocity v'',

$$\Lambda(v)\Lambda(v') = \Lambda(v''),$$

   where the diagonal elements must be equal as in all lambda-matrices. Show that the combination

$$v\gamma/\delta = \text{universal constant} = -c^2,$$

   is one and the same for all inertial frames.

   (e) Prove (6).

   (f) Argue that c, if finite, is the largest relative velocity and the highest velocity of a physical body.

2. (**Obligatory**) Consider the Lorentz transformation for differentials,

$$\begin{pmatrix} dt' \\ dz' \end{pmatrix} = \gamma \begin{bmatrix} 1 & -v/c^2 \\ -v & 1 \end{bmatrix} \begin{pmatrix} dt \\ dz \end{pmatrix}, \tag{7}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, and

   (a) show the the interval

$$ds^2 = c^2dt^2 - dz^2$$

   is invariant under Lorentz transformation.

   (b) show that a moving clock runs slower, than stationary. Hint: consider the transformation of

$$\begin{pmatrix} dt \\ dz = 0 \end{pmatrix}.$$

   (c) show that a moving rod is shorter, than stationary. Hint: consider a transformation into

$$\begin{pmatrix} dt' = 0 \\ dz' \end{pmatrix}.$$