Solutions to Friedman equation

The Friedman equation for a closed universe,

$$d\eta = \pm \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 - 1}}\,. \tag{1}$$

can be integrated for matter-dominated universe, where the pressure is zero, $p = 0$, and the energy density ϵ is equal the mass density μ. The energy conservation equation,

$$d\epsilon = -(\epsilon + p)\frac{3a}{a}\,, \tag{2}$$

gives in this case $\mu a^3 = \text{const}$, which is simply the conservation law of the total mass inside this dust-filled universe.

The volume of the closed universe is $V = 2\pi^2 a^3$ and therefore $\mu = M/V = M/(2\pi^2 a^3)$, where M is the total mass of the universe. Thus const $= M/(2\pi^2)$ and taking the positive square root[1] gives

$$d\eta = \frac{da}{\sqrt{2a_0 a - a^2}} = d\arccos\frac{a_0 - a}{a_0}\,, \tag{3}$$

where $a_0 = \frac{1}{6}\frac{\kappa M}{2\pi^2}$. Integrating first (3) and then $dt = a d\eta$ gives

$$\begin{aligned} a &= a_0(1 - \cos(\eta))\,, \\ t &= a_0(\eta - \sin(\eta))\,. \end{aligned} \tag{4}$$

Thus the life time of a closed universe is finite, $\Delta\eta = 2\pi$, the universe starts with the *Big Bang* at $\eta = 0$ where $a \to 0$ end ends with the *Big Crunch* at $\eta = 2\pi$ where again $a \to 0$.

For the open isotropic universe the Friedman equation reads

$$d\eta = \pm \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 + 1}}\,. \tag{5}$$

A similar integration for the matter-dominated universe with the equation of state $p = 0$, $\epsilon = \mu$ gives

$$\begin{aligned} a &= a_0(\cosh(\eta) - 1)\,, \\ t &= a_0(\sinh(\eta) - \eta)\,. \end{aligned} \tag{6}$$

[1] the $\pm$ sign simply reflects the symmetry of the equation under the substitution $\eta \to -\eta$.

Thus for the open universe the scenario is big-bang $\to$ expansion-forever.

For a flat isotropic universe,

$$ds^2 = dt^2 - b^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{7}$$

the scenario is also big-bang $\to$ expansion-forever[2].

At early stages with high densities the universe was (probably) rather radiation dominated, that is, filled with (noninteracting) photons. The number of photons in the universe is now constant, which for the photon density, n, gives $na^3 = \text{const}$. However, the energy of a photon, $\hbar\omega$, scales with the size of the universe as a^{-1}. Therefore $\epsilon \propto na^{-1}$ and the energy conservation law becomes

$$\epsilon a^4 = \text{const}\,. \tag{8}$$

The increased pressure, however, does not save the universe from the singularity at "the beginning". Indeed, integrating the Friedman equation with $\epsilon a^4 = \mathit{const}$ for early times, $\eta \ll 1$, gives

$$a \propto t^{1/2}\,. \tag{9}$$

Cosmological redshift and Hubble constant

In an isotropic universe the radial ($d\theta = d\phi = 0$) propagation of light is described by

$$0 = ds^2 = a^2(d\eta^2 - d\chi^2)\,, \tag{10}$$

with the solution

$$\chi = \pm\eta + \text{const}\,. \tag{11}$$

Suppose two flashes of light are travelling radially in rapid succession one after another. Their temporal and spatial separations $\Delta\eta = \Delta\chi$ remain constant along their trajectory in the $\{\eta, \chi\}$ coordinates. However, in $\{t, r\}$ the the corresponding separations $\Delta t = a\Delta\eta$ and $\Delta r = a\Delta\chi$ vary with a such that along the light ray $a\omega$ and λ/a remain constant.

Therefore a ray of light with frequency ω_0 emitted at a distance χ and observed at the origin ($\chi = 0$) at time η has the frequency

$$\omega = \omega_0 \frac{a(\eta - \chi)}{a(\eta)} \approx \omega_0(1 - \chi\frac{a'}{a})\,, \tag{12}$$

[2] $\mu b^3 = \text{const}$, $b \propto t^{2/3}$.

that is, redshifted, if the universe expands ($a' > 0$).

The proper distance l to the source of light is $l = \chi a$. Thus the frequency shift z can be written as

$$z \equiv \frac{\omega_0 - \omega}{\omega_0} = \frac{a'}{a^2} l \equiv Hl, \tag{13}$$

where H is the so called Hubble constant,

$$H = \frac{a'}{a^2} = \frac{1}{a}\frac{da}{dt}\,. \tag{14}$$

The current empirical value of the Hubble constant is $H \approx (13.8\,\text{bil. years})^{-1}$.

Inserting $\frac{a'}{a^2} = H$ into Friedman equation leads to

$$\frac{1}{a^2} = H^2 - \frac{\kappa\mu}{3} \tag{15}$$

for a closed matter-dominated universe, and to

$$\frac{1}{a^2} = \frac{\kappa\mu}{3} - H^2 \tag{16}$$

for an open universe.

For the critical density μ_c, such that

$$\frac{\kappa\mu_c}{3} = H^2\,, \tag{17}$$

the universe is flat.

The current measurements show that the relative density $\Omega = \frac{\mu}{\mu_c}$ is close to one with an error about few per cent (flatness problem). About 30% of it is "dark matter"and about 70% is "dark energy". The visible matter constitutes only about 3% of the density.

Exercises

1. Interpret the cosmological red shift $\frac{\omega_0 - \omega}{\omega_0} = Hl$ (l is the distance to the red-shifted galaxy) as a Doppler effect and calculate the velocity with which a galaxy appears to be moving relative to the observer.

2. Show that $\epsilon a^4 = \text{const}$ corresponds to the equation of state $p = \frac{\epsilon}{3}$.