Friedman universe

The *Friedman universe* is a cosmological model where the universe is assumed to be homogeneous, isotropic, and filled with *ideal fluid*. The assumption of homogeneity and isotropicity of the universe is often referred to as the *cosmological principle*. Empirically, it seems to by justified on scales larger that 100 Mpc.

The *Friedman equation* is the Einstein equation applied to a Friedman universe as a whole. It describes the temporal evolution of a Friedman universe.

Spaces with constant curvature

A homogeneous and isotropic universe is (apparently) a space with constant curvature.

Two-dimensional spaces of constant curvature are three-dimensional sphere (positive curvature), pseudo-sphere (negative curvature), and plane (zero curvature).

On the sphere the length element in spherical coordinates is given as

$$dl^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \;, \tag{1}$$

where a is the radius of the sphere.

Let us introduce on the sphere the polar coordinates $\{r, \phi\}$, with r measuring the distance to the north pole. The length of a circle around north pole, $\theta = \text{const}$, is equal $2\pi a \sin\theta$. Therefore if we want the circumference of the circle to be equal $2\pi r$, we need to define $r = a\sin\theta$. Then the length element in the polar coordinates becomes

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 d\phi^2 \,. \tag{2}$$

On a pseudo-sphere, where the curvature is negative, the length element is correspondingly

$$dl^2 = \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2 d\phi^2 \,. \tag{3}$$

In angular coordinates $r = a\sinh\theta$ the latter becomes

$$dl^2 = a^2\left(d\theta^2 + \sinh^2\theta d\phi^2\right) \;. \tag{4}$$

On a plane, where the curvature is zero, the length element is

$$dl^2 = dr^2 + r^2 d\phi^2 \,. \tag{5}$$

Analogously, a three-dimensional space with constant curvature can have one of the following three possible geometries:

- *flat* (zero curvature),

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,; \tag{6}$$

- *closed* (positive curvature),

$$\begin{aligned} dl^2 &= \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= a^2\left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \,, \end{aligned} \tag{7}$$

where $r = a\sin\chi$, $\chi \in [0, \pi]$;

- and *open* (negative curvature),

$$\begin{aligned} dl^2 &= \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= a^2\left(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \,, \end{aligned} \tag{8}$$

where $r = a\sinh\chi$, $\chi \in [0, \infty[$.

Friedman equation

Friedman metric[1] is a (generic) synchronous metric in a homogeneous and isotropic universe,

$$ds^2 = dt^2 - a(t)^2 dl^2 \,, \tag{9}$$

where dl^2 is the line element in a three-dimensional space of constant curvature, and $a(t)$ is the time-dependent *scale factor* of the universe.

Closed universe. In a closed Friedman universe the metric is

$$ds^2 = a^2\left(d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right) \,, \tag{10}$$

where $r = a\sin\chi$, and η is the scaled time coordinate,

$$dt = a d\eta \,. \tag{11}$$

The components of the Ricci tensor are[2]

$$R^\chi_\chi = R^\theta_\theta = R^\phi_\phi = -\frac{1}{a^4}(2a^2 + a'^2 + aa'') \,, \tag{12}$$

$$R^\eta_\eta = \frac{3}{a^4}(a'^2 - aa'') \,, R = -\frac{6}{a^3}(a + a'') \,, \tag{13}$$

[1] also referred to as Friedmann-Lemaitre-Robertson-Walker in different combinations.

[2] $\Gamma^\chi_{\eta\eta} = \Gamma^\theta_{\eta\eta} = \Gamma^\phi_{\eta\eta} = 0$, $\Gamma^\eta_{\eta\eta} = \Gamma^\chi_{\eta\chi} = \Gamma^\theta_{\eta\theta} = \Gamma^\phi_{\eta\phi} = \frac{a'}{a}$, …

where prime denotes the η-derivative.

The energy-momentum tensor for the perfect fluid is

$$T_{ab} = (\epsilon + p)u_a u_b - p g_{ab}\,, \qquad (14)$$

where ϵ is the energy density and p is the pressure. In (synchronous) Friedman coordinates the matter is at rest and the 4-velocity is $u^b = \{\frac{1}{a}, 0, 0, 0\}$.

Correspondingly, the Einstein equation

$$R^a_b - \frac{1}{2}R\delta^a_b = \kappa T^a_b \qquad (15)$$

then has the ${}^\eta_\eta$ component

$$\frac{3}{a^4}(a^2 + a'^2) = \kappa\epsilon, \qquad (16)$$

called the *Friedman equation* for a closed universe, and the three identical spatial equations

$$\frac{1}{a^4}(a^2 + 2aa'' - a'^2) = -\kappa p\,. \qquad (17)$$

If the relation between ϵ and p, called the *equation of state* of the matter, is known, the energy density ϵ can be determined as function of a from the energy conservation equation[3],

$$d\epsilon = -(\epsilon + p)\frac{3da}{a}\,. \qquad (18)$$

Then the solution to the Friedman equation can be obtained as the integral

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 - 1}}\,. \qquad (19)$$

Open universe. In an open Friedman universe the metric is

$$ds^2 = a^2\left(d\eta^2 - d\chi^2 - \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right)\,, \qquad (20)$$

where $r = a\sinh\chi$, and $ad\eta = dt$. This metric can be obtained from the closed universe metric (10) by a formal substitution

$$\{a, \eta, \chi\} \to \{ia, i\eta, i\chi\}\,. \qquad (21)$$

Therefore the Friedman equation for an open universe can be readily obtained from (16) by the substitution (21),

$$\frac{3}{a^4}(a'^2 - a^2) = \kappa\epsilon, \qquad (22)$$

with the integral solution,

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 + 1}}\,. \qquad (23)$$

[3] obtained from $dE = -pdV$

Exercises

1. (**Obligatory**) Consider a flat (Euclidean) isotropic universe with the metric

$$ds^2 = dt^2 - b^2(t)(dx^2 + dy^2 + dz^2)\,.$$

   (a) Calculate the Christoffel symbols. [$\Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} = \frac{b'}{b}$, $\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = bb'$]

   (b) Calculate the Ricci tensor and the Ricci scalar. [$R^t_t = -3\frac{b''}{b}$, $R^x_x = R^y_y = R^z_z = -\frac{b''}{b} - 2\frac{b'^2}{b^2}$]

   (c) Write down the ${}^t_t$ component of the Einstein equation (with perfect fluid). [$3\frac{b'^2}{b^2} = \kappa\epsilon$]

   (d) Write down the energy conservation equation $\frac{dV}{V} = -\frac{d\epsilon}{(\epsilon+p)}$. [$3\ln(b) = -\int\frac{d\epsilon}{(\epsilon+p)}$]

   (e) Integrate the equations for a matter dominated universe ($p = 0$, $\epsilon = \mu$). [$\mu b^3 = \text{const}, b = \text{const}\cdot t^{2/3}$]

   (f) Integrate the equations for a radiation dominated universe ($p = \frac{\epsilon}{3}$). [$\epsilon b^4 = \text{const}, b = \text{const}\cdot t^{1/2}$]

2. Calculate the volumes of the closed and open universes.