Classical tests of GTR

Mercury perihelion advance

In the 19th century it was discovered that interplanetary perturbations cannot account fully for the turning rate of the Mercury's orbit. About 43 arc-seconds per century remained unexplained. The general theory of relativity exactly accounts for this discrepancy.

The Newtonian equation for the equatorial motion of a planet around a star with mass M,

$$u'' + u = \frac{M}{J^2}, \tag{1}$$

where $u = \frac{1}{r}$, $u' = \frac{du}{d\phi}$, and J is a constant, has a periodic elliptic solution with an angular period 2π,

$$u = A\sin(\phi - \phi_0) + \frac{M}{J^2}\,, \tag{2}$$

where A and ϕ_0 are constants. The corresponding relativistic equation,

$$u'' + u = \frac{M}{J^2} + 3Mu^2\ , \tag{3}$$

has an additional relativistic term $3Mu^2$ which causes the perihelion to shift. The small correction, ϵ, to the angular frequency can be found by searching for a solution in the form

$$u = A\sin\left[(1+\epsilon)\phi\right] + B\,, \tag{4}$$

where $\epsilon \ll 1$ and A, B are constants. Setting this into the equation and collecting lowest order terms with $\sin\left[(1+\epsilon)\phi\right]$ gives

$$-A2\epsilon\sin\left[(1+\epsilon)\phi\right] = 3M2AB\sin\left[(1+\epsilon)\phi\right]\ , \tag{5}$$

which gives $\epsilon = -\frac{3M^2}{J^2}$.

The angular distance between two perihelia, ϕ_p, is determined by the equation $(1+\epsilon)\phi_p = 2\pi$, which gives (in lowest order in ϵ) $\phi_p = 2\pi - 2\pi\epsilon$. Correspondingly the shift of the orbit is $\Delta\phi = 2\pi\frac{3M^2}{J^2}$. This accounts precisely for the unexplained advance of the orbit.

Bending of light

General relativity predicts apparent bending of light rays passing through gravitational fields. The bending was first observed in 1919 by A.S. Eddington during a total eclipse when stellar images near the occulted disk of the Sun appeared displaced by some arc-seconds from their usual locations in the sky. Later more precise experiments have unambiguously shown that the amount of deflection agrees with the prediction of general relativity. The *Einstein ring* is an example of the deflection of light from distant galaxies by nearby objects.

In the Newtonian theory the light rays travel along straight lines described by the equation $u'' + u = 0$ with the (straight-line) solution $u = A\sin(\phi - \phi_0)$. The corresponding relativistic equation,

$$u'' + u = 3Mu^2\,, \tag{6}$$

has an additional term, $3Mu^2$, which causes the trajectory of light to deflect from the straight line. Searching for the solution in the form $u = A\cos\phi + \epsilon(\phi)$, where $\epsilon(\phi)$ is a small correction, gives

$$\epsilon'' + \epsilon = 3MA^2\cos^2\phi\,. \tag{7}$$

Assuming $\epsilon(\phi) = C\cos^2\phi + D$, where C and D are constants, gives $\epsilon = MA^2(2-\cos^2\phi)$. The incoming and outgoing rays (where $r = \infty$ and $u = 0$) correspond to the angles ϕ_0 which are the solutions to the equations $u(\phi_0) = 0$. Searching for the solution perturbatively in the form $\phi_0 = \pi/2 + \delta\phi$ gives $\delta\phi = 2MA$.

Thus the angle of deflection between the in-going and out-going rays is $\Delta\phi = 2\delta\phi = 4MA = \frac{4M}{r_0}$ where r_0 is the closest distance between the ray and the central body.

Gravitational red-shift.

Gravitational red shift is a change of the frequency of the electro-magnetic radiation as it passes through a gravitational field. It is a direct consequence of the equivalence principle.

The connection between the proper time interval $\Delta\tau$ and the world time interval Δt (here we only consider stationary gravitational fields where such world time can be introduced) is

$$\Delta\tau = \sqrt{g_{00}}\Delta t\,. \tag{8}$$

Since frequencies are inversely proportional to the time intervals the corresponding connection between world frequency ω_0 and the locally measured frequency ω is $\omega = \frac{\omega_0}{\sqrt{g_{00}}}$. In a weak gravitational field $g_{00} = 1 + 2\phi$ and therefore $\omega = \omega_0(1 - \phi)$. A photon emitted from a point with ϕ_1 and received at a point with ϕ_2 will be shifted by

$$\Delta\omega = (\phi_1 - \phi_2)\omega\,. \tag{9}$$

The famous experiment which verified the gravitational red-shift is generally called the Pound-Rebka-Snider experiment where the Mössbauer effect was used to accurately measure the change of frequency of a photon travelling upwards 22 m in the Earth's field.

Exercises

1. (**Obligatory**) Derive the Kepler's law (the relation between the orbit's period and the radius) for a circular orbit in Schwarzschild metric.

   Answer: like in Newtonian theory, $\omega^2 = M/r^3$.

   Hint: the period is equal $2\pi/\omega$, where $\omega = d\phi/dt$ is the angular frequency which can be found from the geodesics $Du^r = 0$.

2. Show that in a synchronous reference frame ($ds^2 = d\tau^2 + g_{\alpha\beta}dx^\alpha dx^\beta$, where $\alpha, \beta = 1, 2, 3$) the time lines are geodesics.

3. Gravitational waves. In a weak gravitational field the metric tensor g_{ab} is equal to the flat metric η_{ab} plus a small term $h_{ab} : g_{ab} = \eta_{ab} + h_{ab}$. Show the the Riemann tensor to the lowest order in h_{ab} is

$$R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}).$$

   Show, that if coordinates satisfy the condition[1] $(h^a_b - \frac{1}{2}h\delta^a_b)^{,b} = 0$, the Ricci tensor is simplifies to $R_{ab} = -\frac{1}{2}h^{,c}_{ab,c}$. Show that the vacuum Einstein equation now turns into the ordinary wave equation $(\frac{\partial^2}{\partial t^2} - \Delta)h_{ab} = 0$.

[1] $h \equiv h^a_a$