Prologue

General relativity is a classical relativistic theory of gravitation published by Albert Einstein in 1916. It is the accepted description of gravitation in modern physics.

General relativity is a geometric theory where gravitational field is not a material field but rather a curvature of space-time: massive bodies distort space-time in their vicinity which affects the motion of other bodies.

General relativity satisfies the correspondence principle[1]: in the limit of no gravitational fields general relativity reduces to special relativity, and in the limit of weak gravitational fields and non-relativistic velocities it reduces to Newtonian gravitation.

Although not the only relativistic theory of gravitation, general relativity is the simplest theory consistent with experimental data.

General relativity has important astrophysical implications and is a basis of current cosmological models of the universe.

Unlike classical electrodynamics general relativity has not been quantised: a complete and self-consistent theory of quantum gravity does not exist yet.

Equivalence principle

General relativity is based on the Einstein's equivalence principle that postulates the local equivalence between inertial and gravitational forces.

Inertial forces

In an inertial frame a free body moves without acceleration, and its (non-relativistic) equation of motion in Cartesian coordinates is

$$\dot{\vec{v}} = 0, \tag{1}$$

where $\vec{v}$ is the velocity of the body and the dot denotes time derivative. This is an equation for a line (a straight curve): in inertial frames free bodies move without acceleration along lines.

[1]a new theory should reproduce the results of older well-established theories in those domains where the old theories work.

In a non-inertial frame, which moves with acceleration $\vec{a}$ relative the to an inertial frame, the corresponding equation of motion for the free body is

$$\dot{\vec{v}} = -\vec{a}. \tag{2}$$

This is an equation for a (not straight) curve (a parabola, actually): in non-inertial frames free bodies generally move with acceleration along curves.

The system of coordinates, where free bodies move along curves, rather than lines, are called curvilinear.

Equation (2) can be written in the form of the second Newton's law,

$$m\dot{\vec{v}} = \vec{F}_i, \tag{3}$$

where m is the mass of the body, and $\vec{F}_i = -m\vec{a}$ is a fictitious *inertial force* (in this case often called *elevator force*).

Inertial forces have the following properties:

1. Inertial forces are proportional to the masses of the bodies, hence under inertial forces all bodies move with the same acceleration.

2. Inertial forces appear due to geometrical properties of curvilinear coordinate systems, rather than due to some physical fields affecting the body.

3. Inertial forces disappear after a coordinate transformation to an inertial frame.

Equivalence of gravitational and inertial masses

Many experiments attempted to observe the difference between gravitational and inertial masses, including Galileo's measurements of acceleration of balls of different composition rolling down inclined planes, and Newton's measurements of the period of pendulums with different mass but identical length. All experiments reported no observed difference, with the most precise experiment to day (Braginsky and Panov) having the accuracy of one part in a trillion (10^{12}).

These experiments suggest that all bodies under a given gravitational force move with the same acceleration, just like under inertial forces.

Einstein's equivalence principle

Einstein has postulated that all properties of inertial forces are locally[2] fulfilled for gravitational forces. In other words, gravitational forces are unlike other physical forces but much like geometrical inertial forces. This is called the Einstein's equivalence principle. It can be formulated as:

1. Gravitational forces are locally equivalent to inertial forces.
2. An accelerated frame is locally equivalent to a frame in a gravitation field.
3. Gravitational field is locally equivalent to a non-inertial frame.
4. In free fall the effects of gravity disappear in all possible local experiments and general relativity reduces locally to special relativity.

The principle is customarily illustrated by two spacecrafts, one on Earth, the other accelerating with the Earth's gravitational acceleration g in the outer space: the observers in these spacecrafts can not determine by doing local experiments inside spacecrafts whether their craft is at rest in a gravitational field or accelerating.

Unlike inertial forces, gravitational forces vanish at large distances from the sources of gravitation and therefore non-local experiments can well distinguish between ficticious and gravitational forces.

Exercises

1. Consider the motion of a particle with charge e and mass m in a constant uniform electric field $\vec{E}$ which is, say, in the direction of x-axis.

   (a) Suppose that at $t = 0$ the particle was at rest, $\vec{v} = 0$, with the coordinate $\vec{r} = 0$. Find $x(t)$.

   (b) Suppose that at $t = 0$ the particle had $\vec{r} = 0$, but $v_x \neq 0$ and $v_y \neq 0$. Find $x(t)$, $y(t)$ and $x(y)$.

   Hint: the equation of motion of a charged particle in an electro-magnetic field $\vec{E}$, $\vec{H}$ is

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H}\right), \tag{4}$$

   where the (relativistic) momentum $\vec{p}$ and the velocity $\vec{v}$ are connected as

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \tag{5}$$

[2] *locally* means within a limited space, where variations of the fields can be neglected.