Schwarzschild metric: a static spherically symmetric solution of vacuum Einstein equations

A general spherically symmetric static metric can be written as

$$ds^2 = Adt^2 - Bdr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

where A and B are functions of only the radius r. Calculating the Christoffel symbols[1] and the Ricci tensor[2] and then solving the vacuum Einstein equations $R_{ab} = 0$ gives[3] the famous Schwarzschild metric

$$ds^2 = \left(1 - \frac{R}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{R}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2)$$

The integration constant R is determined from the Newtonian limit, $R = 2M$ where M is the mass of the central body[4]. It is called gravitaional or Schwarzschild radius.

Motion in the Schwarzschild metric

In the Schwarzschild metric the geodesic equations, $\frac{d}{ds}(g_{ab}u^b) = \frac{1}{2}g_{bc,a}u^bu^c$, for $a = t, \theta, \phi$ are:

$$\frac{d}{ds}\left[\left(1 - \frac{2M}{r}\right)\frac{dt}{ds}\right] = 0 \;; \qquad (3)$$

$$\frac{d}{ds}\left[r^2\frac{d\theta}{ds}\right] = r^2 \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 \;; \qquad (4)$$

$$\frac{d}{ds}\left[r^2\sin^2\theta\frac{d\phi}{ds}\right] = 0 \;. \qquad (5)$$

Instead of the r-geodesic we shall divide the expression for the Schwarzschild metric (2) by ds^2:

$$1 = \left(1 - \tfrac{2M}{r}\right)\left(\tfrac{dt}{ds}\right)^2 - \left(1 - \tfrac{2M}{r}\right)^{-1}\left(\tfrac{dr}{ds}\right)^2 - r^2\left[\left(\tfrac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\tfrac{d\phi}{ds}\right)^2\right] \qquad (6)$$

The first three equations can be integrated as

$$\theta = \frac{\pi}{2}\;, \quad r^2\frac{d\phi}{ds} = J\;, \quad \left(1 - \frac{2M}{r}\right)\frac{dt}{ds} = E\;, \qquad (7)$$

where J and E are constants. The fourth equation then becomes

$$1 = \frac{E^2}{1 - \frac{2M}{r}} - \frac{\frac{J^2}{r^4}r'^2}{1 - \frac{2M}{r}} - \frac{J^2}{r^2}\;, \qquad (8)$$

where $r' \equiv \frac{dr}{d\phi}$. Traditionally one makes a variable substitution $r = 1/u$,

$$(1 - 2Mu) = E^2 - J^2u'^2 - J^2u^2(1 - 2Mu), \qquad (9)$$

and then differentiates the equation once. Assuming $u' \neq 0$ this gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2\;. \qquad (10)$$

In this form the last term is a relativistic correction to the othewise non-relativistic equation.

The light rays travel along the null-geodesics where $ds^2 = 0$. Consequently instead of ds one needs to use some parameter $d\lambda$ in the geodesic equations $\frac{Dk^a}{d\lambda} = 0$, where $k^a = \frac{dx^a}{d\lambda}$ and also the unity in the left-hand side of equation (6) has to be substituted with zero. This apparently leads to the equation

$$u'' + u = 3Mu^2\;, \qquad (11)$$

which describes the trajectory of a ray of light in the Schwarzschild metric.

Exercises

1. Consider a non-relativistic equatorial ($\theta = \pi/2$) motion of a planet with mass m around a star with mass M discribed by a Lagrangian[5]

   $$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{mM}{r}.$$

   Write down the Euler-Lagrange equations

   $$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}\;, \quad q = r, \phi\;.$$

   Using the first integral $r^2\dot{\phi} = J$ rewrite the r-equation as an equation for the function $u(\phi)$, where $u = 1/r$, and compare with (10).

2. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. Answer: $r = \frac{3}{2}(2M)$.

[1] $\Gamma^r_{rr} = \frac{1}{2}\frac{B'}{B}$, $\Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A}$, $\Gamma^r_{tt} = \frac{1}{2}\frac{A'}{B}$, $\Gamma^\theta_{\theta r} = \frac{1}{r}$, $\Gamma^r_{\theta\theta} = -\frac{r}{B}$, $\Gamma^\phi_{\phi r} = \frac{1}{r}$, $\Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B}$, $\Gamma^\phi_{\phi\theta} = \cot\theta$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$.

[2] $R_{tt} = \frac{A''}{2B} + \frac{A'}{B}(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A})$, $R_{\theta\theta} = 1 - (\frac{r}{B})' - \frac{1}{2}(\frac{A'}{A} + \frac{B'}{B})\frac{r}{B}$, $R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}$.

[3] Making a linear combination $BR_{tt} + AR_{rr} = 0$ gives $A'B + AB' = 0 \;\Rightarrow\; \frac{A'}{A} + \frac{B'}{B} = 0 \;\Rightarrow\; AB = 1$. Then $R_{\theta\theta} = 0$ gives $B = \frac{1}{1-\frac{R}{r}}$, $A = 1 - \frac{R}{r}$, where R is an integration constant.

[4] in the units where $G = 1$

[5] $\dot{A} \equiv \frac{\partial A}{\partial t}$