

Schwarzschild metric: a static spherically symmetric solution of vacuum Einstein equations

A general spherically symmetric static metric can be written as

$$ds^2 = A dt^2 - B dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where A and B are functions of only the radius r . Calculating the Christoffel symbols¹ and the Ricci tensor² and then solving the vacuum Einstein equations $R_{ab} = 0$ gives³ the famous Schwarzschild metric

$$ds^2 = \left(1 - \frac{R}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{R}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2)$$

The integration constant R is determined from the Newtonian limit, $R = 2M$ where M is the mass of the central body⁴. It is called gravitaional or Schwarzschild radius.

Motion in the Schwarzschild metric

In the Schwarzschild metric the geodesic equations, $\frac{d}{ds}(g_{ab}u^b) = \frac{1}{2}g_{bc,a}u^b u^c$, for $a = t, \theta, \phi$ are:

$$\frac{d}{ds} \left[\left(1 - \frac{2M}{r}\right) \frac{dt}{ds} \right] = 0; \quad (3)$$

$$\frac{d}{ds} \left[r^2 \frac{d\theta}{ds} \right] = r^2 \sin\theta \cos\theta \left(\frac{d\phi}{ds} \right)^2; \quad (4)$$

$$\frac{d}{ds} \left[r^2 \sin^2\theta \frac{d\phi}{ds} \right] = 0. \quad (5)$$

Instead of the r -geodesic we shall divide the expression for the Schwarzschild metric (2) by ds^2 :

$$1 = \left(1 - \frac{2M}{r}\right) \left(\frac{dt}{ds}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{ds}\right)^2 - r^2 \left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 \right] \quad (6)$$

The first three equations can be integrated as

$$\theta = \frac{\pi}{2}, \quad r^2 \frac{d\phi}{ds} = J, \quad \left(1 - \frac{2M}{r}\right) \frac{dt}{ds} = E, \quad (7)$$

¹ $\Gamma_{rr}^r = \frac{1}{2} \frac{B'}{B}$, $\Gamma_{ir}^t = \frac{1}{2} \frac{A'}{A}$, $\Gamma_{tt}^r = \frac{1}{2} \frac{A'}{B}$, $\Gamma_{\theta r}^\theta = \frac{1}{r}$, $\Gamma_{\theta\theta}^r = -\frac{r}{B}$, $\Gamma_{\phi r}^\phi = \frac{1}{r}$, $\Gamma_{\phi\phi}^r = -\frac{r \sin^2\theta}{B}$, $\Gamma_{\phi\theta}^\phi = \cot\theta$, $\Gamma_{\phi\phi}^\theta = -\sin\theta \cos\theta$.

² $R_{tt} = \frac{A''}{2B} + \frac{A'}{B} \left(\frac{1}{r} - \frac{B'}{4B} - \frac{A'}{4A}\right)$, $R_{\theta\theta} = 1 - \left(\frac{r}{B}\right)' - \frac{1}{2} \left(\frac{A'}{A}\right) + \frac{B'}{B} \frac{r}{B}$, $R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{B'}{rB}$.

³ Making a linear combination $B R_{tt} + A R_{rr} = 0$ gives $A'B + AB' = 0 \Rightarrow \frac{A'}{A} + \frac{B'}{B} = 0 \Rightarrow AB = 1$. Then $R_{\theta\theta} = 0$ gives $B = \frac{1}{1 - \frac{R}{r}}$, $A = 1 - \frac{R}{r}$, where R is an integration constant.

⁴ in the units where $G = 1$

where J and E are constants. The fourth equation then becomes

$$1 = \frac{E^2}{1 - \frac{2M}{r}} - \frac{\frac{J^2}{r^4} r'^2}{1 - \frac{2M}{r}} - \frac{J^2}{r^2}, \quad (8)$$

where $r' \equiv \frac{dr}{d\phi}$. Traditionally one makes a variable substitution $r = 1/u$,

$$(1 - 2Mu) = E^2 - J^2 u'^2 - J^2 u^2 (1 - 2Mu), \quad (9)$$

and then differentiates the equation once. Assuming $u' \neq 0$ this gives

$$u'' + u = \frac{M}{J^2} + 3Mu^2. \quad (10)$$

In this form the last term is a relativistic correction to the otherwise non-relativistic equation.

The light rays travel along the null-geodesics where $ds^2 = 0$. Consequently instead of ds one needs to use some parameter $d\lambda$ in the geodesic equations $\frac{Dk^a}{d\lambda} = 0$, where $k^a = \frac{dx^a}{d\lambda}$ and also the unity in the left-hand side of equation (6) has to be substituted with zero. This apparently leads to the equation

$$u'' + u = 3Mu^2, \quad (11)$$

which describes the trajectory of a ray of light in the Schwarzschild metric.

Exercises

1. Consider a non-relativistic equatorial ($\theta = \pi/2$) motion of a planet with mass m around a star with mass M discribed by a Lagrangian⁵

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{mM}{r}.$$

Write down the Euler-Lagrange equations

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \quad q = r, \phi.$$

Using the first integral $r^2 \dot{\phi} = J$ rewrite the r -equation as an equation for the function $u(\phi)$, where $u = 1/r$, and compare with (10).

2. Show that a light ray can travel around a massive star in a circular orbit much like a planet. Calculate the radius (in Schwarzschild coordinates) of this orbit. Answer: $r = \frac{3}{2}(2M)$.

⁵ $\dot{A} \equiv \frac{\partial A}{\partial t}$