Newtonian limit: slow motion in weak fields

Newtonian gravitation

The Newton's law of gravitation states that two bodies with masses m and M located at a relative distance r attract each other with the force

$$F = G\frac{mM}{r^2}, \tag{1}$$

where $G \approx 6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$ is the gravitational constant measured by Cavendish.

In other words a distribution of masses with density μ creates a gravitational potential $\phi(\vec{r})$ which satisfies the Poisson equation

$$\Delta\phi = 4\pi G\mu, \tag{2}$$

and a test body m in the gravitational potential ϕ is affected by a force

$$\vec{F} = -m\vec{\nabla}\phi. \tag{3}$$

The last equation can be cast into a variational form with the action

$$\begin{aligned} S &= \int dt\left(-mc^2 + \frac{1}{2}mv^2 - m\phi\right) \\ &= -mc\int dt\left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right). \end{aligned} \tag{4}$$

Comparing with $S = -mc\int ds$ we get (in the limit $c \to \infty$)

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2dt^2 - d\vec{r}^2. \tag{5}$$

In other words in the Newtonian limit the metric tensor can be approximated[1] by $g_{ab} = \eta_{ab} + h_{ab}$ (where η_{ab} is the Minkowski metric tensor and h_{ab} is a small correction) where the g_{00} component is

$$g_{00} = 1 + \frac{2\phi}{c^2}. \tag{6}$$

Newtonian limit of general relativity

In this limit all fields are weak and all velocities are small. Only the $_{00}$ component of the energy-momentum tensor is non-vanishing, $T_{00} = \mu$ where μ is the mass density of the matter. Therefore we shall only consider the $_{00}$ component of the Einstein's equation, $R_{00} = \kappa(T_{00} - \frac{1}{2}g_{00}T)$.

In the Newtonian limit $g_{00} = 1 + 2\phi$, $\Gamma^{\alpha}_{00} = -\phi^{,\alpha}$, $R_{00} = -\phi^{,\alpha}_{,\alpha} = \Delta\phi$, where $\alpha = 1, 2, 3$. The Einstein equation thus turns into the the Poisson's equation $\Delta\phi = \frac{1}{2}\kappa\mu$ which is equivalent to the Newtonian theory if we put $\kappa = \frac{8\pi G}{c^4}$.

Gravitational waves.

In a weak gravitational field the space-time is almost flat and the metric tensor g_{ab} is equal to the flat metric η_{ab} plus a small correction h_{ab}, $g_{ab} = \eta_{ab} + h_{ab}$. The Riemann tensor to the lowest order in h_{ab} is

$$R_{abcd} = \frac{1}{2}(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}). \tag{7}$$

If we choose coordinates such that $(h^a_b - \frac{1}{2}h\delta^a_b)^{,b} = 0$, the Ricci tensor is simply

$$R_{ab} = -\frac{1}{2}h^{,c}_{ab,c} \tag{8}$$

and the vacuum Einstein's equations turn into the ordinary wave equations

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)h_{ab} = 0. \tag{9}$$

The intensity of gravitational radiation by a system of slowly moving bodies is determined by its quadrupole moment $D_{\alpha\beta}$

$$-\frac{dE}{dt} = \frac{G}{45c^5}(D'''_{\alpha\beta})^2 \tag{10}$$

Exercises

1. Calculate the energy-momentum tensor T_{ab} for a particle of mass m with the action $S = -m\int ds$. Hint: calculate the variation of the action with respect to δg_{ab} and represent it in the form $\delta S = -\frac{1}{2}\int T^{ab}\delta g_{ab}ds$.
2. In the Minkowski space consider a scalar field φ with action

   $$S = \int d\Omega\left(-\frac{1}{2}\varphi^{,a}\varphi_{,a} - \frac{1}{2}m^2\varphi^2\right)$$

   and calculate its "translation-invariance" energy-momentum tensor,

   $$T^a_b = \frac{\partial\mathcal{L}}{\partial\varphi_{,a}}\varphi_{,b} - \mathcal{L}\delta^a_b .$$

   Rewrite the action in a generally covariant form and calculate its "metric-derivative" energy-momentum tensor,

   $$\frac{1}{2}\sqrt{-g}T_{ab} = \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{ab}} .$$

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[1] where we have neglected the terms $g_{\alpha\beta}$, $\alpha\beta = 1, 2, 3$ since their contribution to ds^2 is not multiplied by c^2 and is thus neglidible compared to the contribution from g_{00}.