Action for matter and gravitation

Covariant volume element

In a curved space the volume element $d\Omega \equiv d^4x$ has to be substituted with a covariant expression, $\sqrt{-g}d\Omega$, where g is the determinant of the metric tensor g_{ab} $(g<0)$. Indeed the metric tensor transforms as

$$g_{ab} = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b}g'_{cd}. \tag{1}$$

Taking determinant of both sides gives $g = J'^2 g'$, or

$$\sqrt{-g} = J'\sqrt{-g'} \tag{2}$$

where $J' = \left|\frac{\partial x'^a}{\partial x^b}\right|$ is the Jacobian of the transformation. The 4-volume transforms as

$$d\Omega = \left|\frac{\partial x^a}{\partial x'^b}\right| d\Omega' = \frac{1}{J'}d\Omega'. \tag{3}$$

Apparently the combination $\sqrt{-g}d\Omega$ transforms as

$$\sqrt{-g}d\Omega = J'\sqrt{-g'}\frac{1}{J'}d\Omega' = \sqrt{-g'}d\Omega', \tag{4}$$

and is thus a covariant volume element.

Action for matter

The action for matter in general relativity is the good old action from special relativity, only rewritten, if needed, in a generally covariant way. Particularly, $d\Omega \to \sqrt{-g}d\Omega$, $\partial^a\varphi \to g^{ab}\partial_b\varphi$, and $\partial_a A^b \to D_a A^b$. For example,

$$\int (-A_a j^a)\, d\Omega \to \int (-A_a j^a)\sqrt{-g}d\Omega\ , \tag{5}$$

$$\int \frac{-1}{2}\partial^a\varphi\partial_a\varphi d\Omega \to \int \frac{-1}{2}g^{ab}\partial_a\varphi\partial_b\varphi\sqrt{-g}d\Omega \tag{6}$$

$$\int \frac{-1}{16\pi}F^{ab}F_{ab}d\Omega \to \int \frac{-1}{16\pi}F^{ab}F_{ab}\sqrt{-g}d\Omega \tag{7}$$

Energy-momentum tensor of matter

The variation of the matter action, $S_m = \int \mathcal{L}\sqrt{-g}d\Omega$, due to variation δg^{ab} can be written in terms of a symmetric tensor T_{ab},

$$\delta S_m = \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega = -\frac{1}{2}\int T^{ab}\delta g_{ab}\sqrt{-g}d\Omega, \tag{8}$$

where

$$\frac{1}{2}\sqrt{-g}T_{ab} = \frac{\delta(\sqrt{-g}L)}{\delta g^{ab}}. \tag{9}$$

This tensor satisfies the equation $T^{ab}_{;b} = 0$ which in a flat space turns into the energy-momentum conservation equation $T^{ab}_{,b} = 0$ and we thus assume that it is the energy-momentum tensor (see also the exercises next week).

Hilbert's action and Einstein's equation

The Hilbert action for the gravitation,

$$S_g = -\frac{1}{2\kappa}\int R\sqrt{-g}d\Omega, \tag{10}$$

where κ is a (coupling) constant, is the one which leads to Einsteins field equations. Its variation with respect to δg^{ab} is

$$\begin{aligned}\delta\int R\sqrt{-g}d\Omega &= \delta\int g^{ab}R_{ab}\sqrt{-g}d\Omega = \\ &= \int \big(R_{ab}\sqrt{-g}\delta g^{ab} + R_{ab}g^{ab}\delta\sqrt{-g} + \\ &\qquad + g^{ab}\sqrt{-g}\delta R_{ab}\big)d\Omega. \end{aligned}\tag{11}$$

The last term can be proved (see t'Hooft) to not contribute; in the second term we have[1]

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}gg^{ab}\delta g_{ab} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}. \tag{12}$$

Thus the variation of the Hilbert action is

$$\delta S_g = -\frac{1}{2\kappa}\int\left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\delta g^{ab}\sqrt{-g}d\Omega. \tag{13}$$

Combining (8) and (13) into $\delta(S_m + S_g) = 0$ gives the famous Einstein's equation,

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}\ . \tag{14}$$

Exercises[2]

1. In a curved space the electromagnetic field strength tensor F_{ab} is defined as $F_{ab} = A_{b;a} - A_{a;b}$ and the first Maxwell equation is $F_{ab;c} + F_{bc;a} + F_{ca;b} = 0$. Show that in the torsion free space of general relativity, $\Gamma^a_{bc} = \Gamma^a_{cb}$, these equations can still be written as in Minkowski space, $F_{ab} = A_{b,a} - A_{a,b}$ and $F_{ab,c} + F_{bc,a} + F_{ca,b} = 0$.

2. Derive the second Maxwell equation in a curved space,
$$\left(\sqrt{-g}F^{ab}\right)_{,a} = 4\pi\sqrt{-g}j^b\ ,$$
from the action
$$\int\left(-\frac{1}{16\pi}F^{ab}F_{ab} - A_a j^a\right)\sqrt{-g}d\Omega.$$
Show that the equation can also be written as $F^{ab}_{;a} = 4\pi j^b$, Hints:
   (a) show that $\Gamma^a_{ba} = \frac{1}{2g}g_{,b} = (\ln\sqrt{-g})_{,b}$
   (b) show that $F^{ab}_{;a} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ab})_{,a}$

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[1] $dg = gg^{ab}dg_{ab} = -gg_{ab}dg^{ab}$.
[2] notation: $_{;a} \equiv D_a \equiv \frac{D}{dx^a}$ and $_{,a} \equiv \partial_a \equiv \frac{\partial}{dx^a}$