The Riemann curvature tensor

The Christoffel symbols are not covariant tensors and cannot be easily used in variational principles. In this section we shall find a covariant tensor associated with the curvature of the space which can be used in constructing actions.

If we make a parallel transform ($DA^a = 0$) of a vector A along a path dx, the components of the vector will generally change, $dA^a = -\Gamma^a_{bc} dx^b A^c$. The change $\Delta A^a = \oint dA^a$ along an infinitesimal closed contour is apparently a covariant quantity. It must be proportional to the vector itself and the area of the surface enclosed by the contour. Let us calculate this change using a simple rectangular contour with sides dx and dx'. The vector A^a after a parallel transform along dx and then dx' turns into

$$A^a(x + dx + dx') = A^a - \Gamma^a_{bc} dx^b A^c - \left(\Gamma^a_{bc} + \Gamma^a_{bc,d} dx^d\right) dx'^b \left(A^c - \Gamma^c_{ef} dx^e A^f\right) . \quad (1)$$

Similarly, after a parallel transform first along dx' and then along dx

$$A^a(x + dx' + dx) = A^a - \Gamma^a_{bc} dx'^b A^c - \left(\Gamma^a_{bc} + \Gamma^a_{bc,d} dx'^d\right) dx^b \left(A^c - \Gamma^c_{ef} dx'^e A^f\right) . \quad (2)$$

Subtracting these two and leaving only the lowest order term gives (after some index renaming)

$$\begin{aligned} \Delta A^a &= A^a(x + dx + dx') - A^a(x + dx' + dx) \\ &= -\left(\Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce}\Gamma^e_{bd} + \Gamma^a_{de}\Gamma^e_{bc}\right) A^b dx^c dx'^d \quad (3) \\ &\equiv -R^a_{bcd} A^b dx^c dx'^d . \end{aligned}$$

The expression in brackets is called the Riemann tensor R^a_{bcd},

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}. \quad (4)$$

It follows from (3) that if the Riemann tensor vanishes, the vector A^a does not change if parallely transported along an arbitrary infinitesimal closed path. Also after a parallel transform from x to $x + \delta x$ the vector $A^a(x + \delta x)$ is independent on which path is taken. The space where Riemann tensor vanishes everywhere is called flat.

The Riemann tensor defines also the commutator of covariant derivatives

$$(D_a D_b - D_b D_a) A_c = R^d_{abc} A_d, \quad (5)$$

where $D_a \equiv \frac{D}{dx^a}$.

Properties of the curvature tensor

(Read about them in your textbook.)

$$\begin{aligned} R_{abcd} &= -R_{bacd}, & (6) \\ R_{abcd} &= -R_{abdc}, & (7) \\ R_{abcd} &= R_{cdab}, & (8) \\ R_{a[bcd]} &= 0, & (9) \\ R_{ab[cd;e]} &= 0, & (10) \end{aligned}$$

where the square brackets denote symmetrization over the indexes and the semi-colon is a covariant derivative. The fourth and fifth identities are sometimes called the "algebraic Bianchi identity" and the "differential Bianchi identity", respectively.

Ricci tensor

$$R_{ab} = R^d_{adb} \quad (11)$$

Ricci scalar

$$R = g^{ab} R_{ab} \quad (12)$$

Exercises

1. Compute all the non-vanishing components of the Riemann tensor R_{abcd} (where $a, b, c, d = \theta, \phi$) for the metric

   $$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (13)$$

   on a 2-dimensional sphere of radius r. Calculate also the Ricci tensor R_{ab} and the scalar curvature (Ricci scalar) R.

   Answer:

   $$R_{\theta\phi\theta\phi} = r^2 \sin^2\theta = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi}$$