Variational principle

Action

For a given trajectory of motion of a physical system the *action*, S, is a suitable covariant scalar integral along the trajectory, The variational principle is then formulated as follows: that trajectory, on which the action has an extremum, is the solution. Or: the solution is the stationary trajectory of the action. Or: the variation of the action vanishes on the solution,

$$\delta S = 0. \tag{1}$$

Example: a free body

The action of a free body with mass m is given by the following covariant integral along the body's trajectory,

$$S = -m \int ds \ . \tag{2}$$

To calculate the variation of the action we first vary the expression $ds^2 = g_{ab} dx^a dx^b$,

$$2ds\delta ds = \delta g_{ab} dx^a dx^b + 2 g_{ab} \delta dx^a dx^b \tag{3}$$

which gives the variation of ds,

$$\delta ds = \frac{1}{2} \delta g_{ab} u^a u^b ds + d\delta x^a u_a. \tag{4}$$

The second term should be integrated by parts using

$$d\delta x^a u_a = d(\delta x^a u_a) - \delta x^c du_c. \tag{5}$$

The full differential does not contribute to the variation, and we finally arrive at[1]

$$\delta S = -m \int ds \delta x^c \left(-\frac{du_c}{ds} + \frac{1}{2} g_{ab,c} u^a u^b \right) = 0 \tag{6}$$

Since the variation δx is arbitrary, it is the expression in brackets that should be equal zero, which gives the equation of motion

$$\frac{du_c}{ds} - \frac{1}{2} g_{ab,c} u^a u^b = 0. \tag{7}$$

which is equivalent to the geodesic equation (see the exercise).

Exercises

1. Derive the Newton's equation of motion

$$m \frac{d\vec{v}}{dt} = -\vec{\nabla} V$$

of a non-relativistic body with mass m, coordinates $\vec{r}$, and velocity $\vec{v}$ in a potential $V(\vec{r})$ using the action

$$S = \int \left(\frac{m\vec{v}^2}{2} - V(\vec{r}) \right) dt.$$

2. In the Minkowski space derive the Lorentz force equation

$$m \frac{du^a}{ds} = eF^{ab} u_b$$

for a relativistic body with mass m and charge e moving in an electromagnetic field $F_{ab} = \partial_a A_b - \partial_b A_a$ using the action

$$S = -mc \int ds - e \int A^a dx_a.$$

3. In the Minkowski space derive the second Maxwell equation, $\partial_a \partial^a A^b = 4\pi j^b$, using the action

$$S = -\frac{1}{8\pi} \int \partial_a A_b \partial^a A^b d^4x - \int A^a j_a d^4 x.$$

Show that with the Lorentz condition, $\partial_a A^a = 0$, it is equivalent to $\partial_a F^{ab} = 4\pi j^a$.

4. In a curved space derive the geodesic equation[2]

$$\frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0,$$

for the motion of a free body in a curvilinear frame using the action

$$S = -mc \int ds,$$

where $ds^2 = g_{ab} dx^a dx^b$.

[1] $f_{,a} \equiv \frac{\partial f}{\partial x^a}$

[2] where $\Gamma_{abc} = \frac{1}{2}(g_{ab,c} - g_{bc,a} + g_{ca,b})$ are the Christoffel symbols.