Constantly Accelerated (Rindler) Frame

The non-inertial Rindler frame (apparently named after Wolfgan Rindler) is made of constantly and uniformly accelerated clocks. The accelerations are tuned such that if two of the clocks are connected with a solid rod, no stresses appear in the rod in the process of motion. In other words it is, in principle, possible to realize the Rindler frame with material bodies. The Rindler frame is very useful as a relativistic model of a constant uniform gravitational field. It is often used as a test-ground for all sorts of theories. We shall see on the example of the Rindler frame that non-inertial frames have some peculiar features: they are curved, they have horizons, time goes differently at different places.

Consider a 1+1 dimensional inertial frame K with coordinates t, x where x-axis is pointing up. The non-inertial frame K' is made of clocks which are constantly accelerating upwards. The clocks are connected with solid rods of unit length. The bottom clock is marked with zero and is accelerating with acceleration g. The next clock is marked with number 1, the next with number 2 and so forth, making a ruler to measure distances. Let us designate the time and distance in this frame as τ, ξ and let us have at $t = 0$ $\tau = 0$ and $x = \xi$.

The relativistic equation of motion of a body accelerating with a constant acceleration a is

$$\frac{d}{dt}\frac{v}{\sqrt{1-v^2}} = a \,, \tag{1}$$

with the solution

$$x = x_0 + \frac{1}{a}\sqrt{1+a^2t^2}. \tag{2}$$

The clock at coordinate ξ accelerates with some acceleration $a(\xi)$. Since there are no stresses in the connecting rods the higher clock has to accelerate a bit slower, more precisely the acceleration must satisfy the equation $da = -a^2 d\xi$ (prove it). The solution is

$$\frac{1}{a} = \xi + \frac{1}{g}, \tag{3}$$

where we have used that $a(\xi = 0) = g$.

Let us choose the time τ to be the proper-time of the clocks. Then the relativistic invariant ds^2 for a given ξ should be equal $d\tau^2$. Calculating $ds^2|_{d\xi=0} = \left(dt^2 - dx^2\right)_{d\xi=0}$ using (2) gives the following transformation rules:

$$\begin{aligned} x &= (\xi + 1/g)\cosh(\frac{\tau}{\xi+1/g}) - 1/g, & (4)\\ t &= (\xi + 1/g)\sinh(\frac{\tau}{(\xi+1/g)}) & (5)\\ ds^2 &= d\tau^2 - \frac{2d\tau d\xi}{(\xi+\frac{1}{g})^2} - (1 + \frac{\tau^2}{(\xi+\frac{1}{g})^2})d\xi^2. & (6)\end{aligned}$$

The frame is apparently curved. Let's simplify it a bit with new coordinates $g\eta = \frac{\tau}{(\xi+1/g)}$ and $\rho = \xi + 1/g$:

$$\begin{aligned} x &= \rho\cosh(g\eta) - 1/g & (7)\\ t &= \rho\sinh(g\eta) & (8)\\ ds^2 &= (g\rho)^2 d\eta^2 - d\rho^2 & (9)\end{aligned}$$

We notice important effects in the accelerated frame:

1. The Rindler coordinates are *curved*, $ds^2 = dt^2 - dx^2 = (g\rho)^2 d\eta^2 - d\rho^2$;
2. The local clock rate in the Rindler frame varies with height, $ds^2|_{d\rho=0} = (g\rho)^2 d\eta^2$;
3. The lines $\eta =$ const converge towards the *event horizons* – the boundaries that separate the part of the total space-time unavailable for the observer in the elevator.

The equivalence principle tells that all these effects must as well be present in gravitational fields.

Exercises

1. In the Rindler space $ds^2 = (1 + g\xi)^2 d\eta^2 - d\xi^2$
   (a) calculate g_{ab}, g^{ab} and Christoffel symbols;
   (b) write down the geodesic equations;
   (c) in the equations $\frac{d^2t}{ds^2} = 0$, $\frac{d^2x}{ds^2} = 0$ make a variable substitution to η, ξ coordinates. Check that the corresponding equations are equivalent to geodesic equations;
   (d) make the non-relativistic limit ($c \to \infty$) and prove, that the geodesic equation reduces to the Newtonian non-relativistic equation $\frac{d^2\xi}{d\eta^2} = -g$.
2. Prove that a constantly accelerating bar with length $d\xi$ must have its ends accelerate differently with a difference $da = -a^2 d\xi$. Hint: since it accelerates, a moment later it will look shorter, thus the lower end must be accelerating faster, than the upper end.
3. Explain the twins paradox from the point of view of the accelerating observer.