Covariant differentiation in curvilinear coordinates

Covariant vectors and metric

A set of four quantities A^a, where $a = 0, 1, 2, 3$, is called a *contravariant vector* if under a transformation of coordinates $x \to x'(x)$ it transforms as coordinate differentials dx^a, while a set of four quantities A_a is called a *covariant vector* if it transforms[1] as derivatives of a scalar $\partial\phi/\partial x^a$,

$$A^a = \frac{\partial x^a}{\partial x'^b}A'^b,\ A_a = \frac{\partial x'^b}{\partial x^a}A'_b. \tag{1}$$

The contraction A^aB_a is clearly invariant under coordinate transformation, $A^aB_a = A'^aB'_a$.

The metric tensor g_{ab} defines the invariant element of curvilinear coordinates, $ds^2 = g_{ab}dx^adx^b$. Since dx^a is an arbitrary contra-variant vector one can conclude that the construction $g_{ab}dx^b$ transforms as a co-variant vector and thus that the metric tensor connects contra- and co-variant vectors, $A_a = g_{ab}A^b$.

Covariant differential

In curvilinear coordinates the differential of a vector dA^a is not a covariant quantity, since generally $dA_a = d(g_{ab}A^b) \neq g_{ab}dA^b$. Indeed in curvilinear coordinates the unit vectors are generally not orthogonal and not normalized, therefore a covariant differential, denoted DA^a, has to contain an additional contribution (see the exercise) which is customarily written through the so-called Christoffel symbols

$$DA^a = dA^a + \Gamma^a_{bc}A^bdx^c\ . \tag{2}$$

Differential of a scalar $d(A_aB^b)$ is already a covariant quantity, therefore $D(A_aB^a) = d(A_aB^b)$ for an arbitrary B^a, which leads to

$$DA_a = dA_a - \Gamma^b_{ac}A_bdx^c\ . \tag{3}$$

For DA_a to be a covariant quantity one needs

$$DA_a = g_{ab}DA^b = Dg_{ab}A^b, \tag{4}$$

i.e. the covariant derivative of the metric tensor[2] has to vanish, $Dg_{ab} = 0$, which defines the Christoffel symbols,

$$\Gamma_{abc} = \frac{1}{2}\left(\frac{dg_{ab}}{dx^c} - \frac{dg_{bc}}{dx^a} + \frac{dg_{ac}}{dx^b}\right). \tag{5}$$

Geodesic as a constant velocity trajectory

A particle in curvilinear coordinates moves in such a way that the covariant derivative of its velocity u^a vanishes, $Du^a = 0$. This leads to the so called *geodesic* equation,

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0. \tag{6}$$

Exercises

1. Prove (5) e.g. by rewriting $Dg_{ab} = 0$ through Christoffel symbols and making a linear combination of three such equations with appropriately renamed indexes.

2. Let x, y be Cartesian coordinates in a flat two-dimensional space with metric $dl^2 = dx^2 + dy^2$. Consider polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ (another notation $x^r \equiv r$, $x^\theta \equiv \theta$)

   (a) From[3] $d\vec{r} = dr\vec{e}_r + d\theta\vec{e}_\theta$ find $\vec{e}_a$, $a = r, \theta$.

   (b) Find $g_{ab} = \vec{e}_a \cdot \vec{e}_b$ $(a, b = r, \theta)$ and check, that it is identical to the tensor from the metric $dl^2 = dx^adx^bg_{ab}$.

   (c) Find $g^{ab} = g^{-1}_{ab}$ and $\vec{e}^a$.

   (d) Consider a vector $\vec{A} = A^a\vec{e}_a$ with a differential

   $$d\vec{A} = dA^a\vec{e}_a + A^a\frac{\partial\vec{e}_a}{\partial x^b}dx^b$$

   and a covariant differential

   $$DA^a \equiv \vec{e}^a \cdot d\vec{A} = dA^a + \left(\vec{e}^a \cdot \frac{\partial\vec{e}_b}{\partial x^c}\right)A^bdx^c.$$

   Prove that

   $$DA_a \equiv \vec{e}_a \cdot d\vec{A} = dA_a - \left(\vec{e}^b \cdot \frac{\partial\vec{e}_a}{\partial x^c}\right)A_bdx^c.$$

   The expression in parentheses is apparently the Christoffel symbol $\Gamma^a_{bc} = \left(\vec{e}^a \cdot \frac{\partial\vec{e}_b}{\partial x^c}\right)$. Calculate it and compare with (5).

   (e) Consider the parametric equations for a straight line in x, y coordinates,

   $$\frac{d^2x}{ds^2} = 0\ ,\ \frac{d^2y}{ds^2} = 0.$$

   Make a variable substitution and obtain the corresponding equations in the r, θ coordinates. Prove that they are identical to (6).

---

[1] in the following $A^aB_a \equiv \sum_a A^aB_a$

[2] considering $D(A^aB^b)$ one can derive the covariant differential of a tensor, $DF^{ab} = dF^{ab} + \Gamma^a_{cd}F^{cb}dx^d + \Gamma^b_{cd}F^{ac}dx^d$.

[3] $d\vec{r} \equiv dx\vec{e}_x + dy\vec{e}_y$, where $\vec{e}_x \cdot \vec{e}_x = \vec{e}_y \cdot \vec{e}_y = 1$, $\vec{e}_x \cdot \vec{e}_y = 0$.