Friedman universe

Geometry of a space with constant curvature

Suppose the universe is uniformly and isotropically filled with matter. The curvature, if any, is then constant througout the universe. The surface of a sphere is apparently a space with constant curvature, where the length element in spherical coordinates is

$$dl^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \quad (1)$$

where a^2 is the sphere's radius. Let us indtroduce polar coordinates $\{r, \phi\}$ on the surface. The length of an surface parallel is equal $2\pi a \sin\theta$, therefore if we the curcumference of a circle to equal $2\pi r$ we need to introduce $r = a\sin\theta$. The length element in our polar coordinates is then

$$dl^2 = \frac{dr^2}{1-\frac{r^2}{a^2}} + r^2 d\phi^2 , \quad (2)$$

which is a general length element of a space with constant positive curvature. Another possibility[1] is

$$dl^2 = \frac{dr^2}{1+\frac{r^2}{a^2}} + r^2 d\phi^2 , \quad (3)$$

which is a general length element of a space with negative constant curvature. In spherical coordinates $r = a\sinh\theta$ the latter is

$$dl^2 = a^2\left(d\theta^2 + \sinh^2\theta d\phi^2\right) , \quad (4)$$

Generalising to three dimensions: the 3D space with constant curvature can have one of the three possible geometries: flat[2]; closed,

$$dl^2 = \frac{dr^2}{1-\frac{r^2}{a^2}} + r^2 d\Omega^2 = a^2\left(d\chi^2 + \sin^2\chi d\Omega^2\right) , \quad (5)$$

where $r = a\sin\chi$; or open,

$$dl^2 = \frac{dr^2}{1+\frac{r^2}{a^2}} + r^2 d\Omega^2 = a^2\left(d\chi^2 + \sinh^2\chi d\Omega^2\right) , \quad (6)$$

where $r = a\sinh\chi$.

Friedman equation and solutions

Friedman metric describes a homogeneous and isotropic universe. For the closed universe the interval is

$$ds^2 = a^2(d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)), \quad (7)$$

[1] the third possibility is a flat space, $a = \infty$

[2] $dl^2 = dx^2 + dy^2 + dz^2$

where $r = a\sin\chi$, η is the scaled time coordinate, $dt = ad\eta$, and $a(\eta)$ is the scale parameter of the universe (the radius of the 4-sphere). The components of the Ricci tensor are

$$R^\chi_\chi = R^\theta_\theta = R^\phi_\phi = -(\frac{1}{a^4})(2a^2 + a'^2 + aa'') , \quad (8)$$

$$R^\eta_\eta = (\frac{3}{a^4})(a'^2 - aa'') , R = -(\frac{6}{a^3})(a + a'') , \quad (9)$$

where prime denotes the derivative with respect to η.

Assuming that the universe is filled with a perfect fluid the energy-momentum tensor of the matter is $T_{ab} = (\epsilon + p)u_a u_b - p g_{ab}$ where ϵ is the energy density and p is the pressure. In our frame, where the matter is at rest, the 4-velosity $u^a = (\frac{1}{a}, 0, 0, 0)$.

The Einstein's equations $R^a_b - \frac{1}{2}R\delta^a_b = \kappa T^a_b$ will then have the ${}^\eta_\eta$ component

$$(\frac{3}{a^4})(a^2 + a'^2) = \kappa\epsilon , \quad (10)$$

called Friedman's equation, and the three identical spatial equations $(\frac{1}{a^4})(a^2 + 2aa'' - a'^2) = -\kappa p$.

The Friedman's equation together with the energy conservation equation, $d\epsilon = -(\epsilon + p)3\frac{da}{a}$, can be integrated for the matter dominated universe, where the pressure is zero, $p = 0$, and the energy density ϵ is equal to the mass density μ. The energy conservation gives $\mu a^3 = Const$ and the subsequent integration of the Friedmann's equation gives for a closed universe ($a^2 > 0$) a "Big Bang → Big Crunch" scenario:

$$a = a_0(1 - \cos(\eta)) , \; t = a_0(\eta - \sin(\eta)) \quad (11)$$

For the open isotropic universe the Friedman's equation provides a "Big Bang → Expansion Forever" scenario:

$$a = a_0(\cosh(\eta) - 1) , \; t = a_0(\sinh(\eta) - \eta) \quad (12)$$

For a flat isotropic universe $ds^2 = dt^2 - b^2(t)(dx^2 + dy^2 + dz^2)$ the scenario is also "Big Bang → Expansion Forever" (see the Exercise): $\mu b^3 = const, b = const\ t^{2/3}$.

At early stages with high densities the universe was (probably) rather radiation dominated, $p = \frac{\epsilon}{3}$. This, however, doesn't save us from the singular point at $\eta = 0$. Indeed, we have (for $\eta \ll 1$) : $\epsilon a^4 = const$, $a = const \cdot t^{1/2}$