Radial fall in the Scwarzschild field

Lemaitre coordinates

In the Schwarzschild metric

$$ds^2 = (1-\frac{r_g}{r})dt^2 - \frac{dr^2}{1-\frac{r_g}{r}} - r^2(d\theta^2+\sin^2\theta d\phi^2), \quad (1)$$

the gravitational radius $r_g = 2M$ is a singular point and inside the gravitational radius r becomes time-like and t becomes space-like. This singularity however is not a physical divergence but rather an artifact of the (false) assumption, that a static Schwarzschild coordinates can be realized under the gravitational radius with material bodies.

A transformation to the Lemaitre coordinates τ, ρ

$$d\tau = dt + \sqrt{\frac{r_g}{r}}\frac{dr}{1-\frac{r_g}{r}}, \quad (2)$$

$$d\rho = dt + \sqrt{\frac{r}{r_g}}\frac{dr}{1-\frac{r_g}{r}} \quad (3)$$

leads to the Lemaitre metric, where the singularity at r_g is removed,[1]

$$ds^2 = d\tau^2 - \frac{r_g}{r}d\rho^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad (4)$$

where $r = [\frac{3}{2}(\rho-\tau)]^{2/3} r_g^{1/3}$, which is obtained by integrating $d\rho - d\tau = \sqrt{\frac{r}{r_g}}dr$.

The Lemaitre coordinates are synchronous[2] and are thus realized by a system of clocks in a free radial fall from infinity towards the origin.

Radiall fall towards the origin

For a free falling body, $d\rho = 0$, equation (3) gives

$$dt = -\sqrt{\frac{r}{r_g}}\frac{1}{(1-\frac{r_g}{r})}dr \ . \quad (5)$$

In the region $r \gtrsim r_g$ we have to the lowest order

$$dt = -\frac{r_g}{r-r_g}dr \ , \quad \Rightarrow \quad \frac{r-r_g}{r_0-r_g} = e^{-\frac{t-t_0}{r_g}} . \quad (6)$$

Apparently, it takes a free falling body infinitely long t-time[3] to reach the Schwarzschild radius.

On the contrary, a free falling Lemaitre clock moves from some given r_0 to the gravitational radius (and also to the origin) within finite τ-time $\Delta\tau$,

$$\Delta\tau = -\int_{r_0}^{r_g}\sqrt{\frac{r}{r_g}}dr = \frac{2}{3}\left(\frac{r_0^{3/2}-r_g^{3/2}}{r_g^{1/2}}\right) . \quad (7)$$

[1] there remains a genuine singularity at the origin
[2] the metric has the form $ds^2 = d\tau^2 - \ldots$
[3] the time used by the outer observer

Event horizon and black holes

Along the trajectory of a radial light ray $ds^2 = 0 = d\tau^2 - \sqrt{\frac{r_g}{r}}d\rho^2$, which gives

$$d\rho = \pm\sqrt{\frac{r}{r_g}}d\tau \ , \quad (8)$$

where plus and minus describe the rays of light sent correspondingly up and down. Inserting $d\rho = d\tau + \sqrt{\frac{r}{r_g}}dr$ into (8) leads to

$$dr = \left(\pm 1 - \sqrt{\frac{r_g}{r}}\right)d\tau. \quad (9)$$

Apparently if $r < r_g$ then there is always $dr < 0$ and thus the light rays emitted radially inwards and outwards both end up at the origin. In other words no signal can escape from inside the gravitational radius – a phenomenon called an event horizon.

Thus a massive object with a size less than the gravitational radius, called a black hole, is completely under the event horizon and its interior is totally invisible. The black holes can be detected if they interact with the matter outside the event horizon.

Excercises

1. What is the escape velocity[4] at a coordinate r in the Schwarzschild field? Investigate the limits $r \ll r_g$ (should be Newtonian) and $r \to r_g$ (should become impossible to escape, right?).
   (a) show that the equation of motion of a free radially moving body in the Schwazschild field can be written as
   $$\left(1-\frac{r_g}{r}\right)\frac{dt}{ds} = E \ , \quad E^2 - \left(\frac{dr}{ds}\right)^2 = 1 - \frac{r_g}{r}.$$
   (b) show that if the body has reached $r = \infty$ the integration constant E is its energy divided by mass;
   (c) show that for a body, starting a free radial fall from infinity with zero velocity, $E = 1$.
2. What is the g-force[5] experienced by an observer at rest at a coordinate $r > r_g$ in the Schwarzschild field? Investigate the limits $r \ll r_g$ (should be Newtonian) and $r \to r_g$ (should be something horrible, I guess).

[4] a body starting a free radial fall at infinity with zero velocity will reach a given r with $\frac{dr}{ds}$ equal escape velocity;
[5] the g-force is equal the proper-acceleration $\frac{d^2r}{ds^2}$ of a free radially falling body at the moment when the body is at rest